Math 33A
Linear Algebra and Applications

Discussion for July 18-21, 2022

Problem 1.

The following determinant was introduced by Alexandre-Theophile Vandermonde. Consider distinct real numbers $a_0, \ldots, a_n$, we define the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{bmatrix}.$$

Vandermonde showed that $\det(A) = \prod_{i>j}(a_i - a_j)$, the product of all differences $a_i - a_j$, where i exceeds j.

(a) Verify this formula in the case of $n = 1$.

(b) Suppose the Vandermonde formula holds for $n-1$. You are asked to demonstrate it for n. Consider the function

$$f(t) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}.$$

Explain why $f(t)$ is a polynomial of n-th degree. Find the coefficient k of t^n using Vandermonde's formula for $a_0, \ldots, a_{n-1}$. Explain why $f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0$. Conclude that $f(t) = k(t-a_0)(t-a_1)\cdots(t-a_{n-1})$ for the scalar k you found above. Substitute $t = a_n$ to demonstrate Vandermonde's formula.

Problem 2($\star$).

Find

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}$$

using Vandermonde's formula and using the usual definition of determinant.

Problem 3.

For n distinct scalars $a_1, \ldots, a_n$, find

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix}.$$

Problem 4.

In his groundbreaking text Ars Magna, the Italian mathematician Gerolamo Cardano explains how to solve cubic equations. In Chapter XI, he considers the following example: $x^3 + 6x = 20$.

(a) Explain why this equation has exactly one (real) solution. Here, this solution is easy to find by inspection. The point of this exercise is to show a systematic way to find it.

(b) Cardano explains his method as follows (we are using modern notation for the variables): "I take two cubes v^3 and u^3 whose difference shall be 20, so that the product vu shall be 2, that is, a third of the coefficient of the unknown x. Then, I say that $v-u$ is the value of the unknown x". Show that if v and u are chosen as stated by Cardano, then $x = v - u$ is indeed the solution of the equation $x^3 + 6x = 20$.

(c) Solve the system

$$\begin{aligned} v^3 - u^3 &= 20 \\ vu &= 2 \end{aligned}$$

to find u and v.

(d) Consider the equation $x^3 + px = q$, where p is positive. Using your previous work as a guide, show that the unique solution of this equation is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Check that this solution can also be written as

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

What can go wrong when p is negative?

(e) Consider an arbitrary cubic equation $x^3 + ax^2 + bx + c = 0$. Show that the substitution $x = t - (a/3)$ allows you to write this equation as $t^3 + pt = q$.

Problem 5.

Consider an $n \times n$ matrix A. A subspace V of $\mathbb{R}^n$ is said to be A-invariant if $A\vec{v}$ is in V for all $\vec{v}$ in V. Describe all the one-dimensional A-invariant subspaces of $\mathbb{R}^n$ in terms of the eigenvectors of A.

Problem 6($\star$).

Consider an arbitrary $n \times n$ matrix A. What is the relationship between the characteristic polynomials of A and A^T ? What does your answer tell you about the eigenvalues of A and A^T?

Problem 7.

Suppose matrix A is similar to B. What is the relationship between the characteristic polynomials of A and B? What does your answer tell you about the eigenvalues of A and B?