Math 33A
Linear Algebra and Applications

Discussion for July 18-21, 2022

Problem 1.

The following determinant was introduced by Alexandre-Theophile Vandermonde. Consider distinct real numbers $a_0, \ldots, a_n$, we define the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_n \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{bmatrix}.$$

Vandermonde showed that $\det(A) = \prod_{i>j}(a_i - a_j)$, the product of all differences $a_i - a_j$, where i exceeds j.

(a) Verify this formula in the case of $n = 1$.

(b) Suppose the Vandermonde formula holds for $n-1$. You are asked to demonstrate it for n. Consider the function

$$f(t) = \det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}.$$

Explain why $f(t)$ is a polynomial of n-th degree. Find the coefficient k of t^n using Vandermonde's formula for $a_0, \ldots, a_{n-1}$. Explain why $f(a_0) = f(a_1) = \cdots = f(a_{n-1}) = 0$. Conclude that $f(t) = k(t-a_0)(t-a_1)\cdots(t-a_{n-1})$ for the scalar k you found above. Substitute $t = a_n$ to demonstrate Vandermonde's formula.

Solution:

(a) For $n = 1$ we have

$$A = \begin{bmatrix} 1 & 1 \\ a_0 & a_1 \end{bmatrix} \quad \text{so} \quad \det(A) = a_1 - a_0$$

and the formula holds.

(b) Suppose that the formula holds for $n-1$, let

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} & t \\ a_0^2 & a_1^2 & \cdots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & & \vdots & \\ a_0^n & a_1^n & \cdots & a_{n-1}^n & t^n \end{bmatrix}$$

and expand down the rightmost column. This yields

$$
\begin{aligned}
f(t) &= \det(B) = \\
&= \sum_{i=0}^{n-1} (-1)^{i+1+n+1} t^i \det(B_{i+1,n+1}) + (-1)^{n+1+n+1} t^n \det(B_{n+1,n+1}) = \\
&= \sum_{i=0}^{n-1} (-1)^{i+n} t^i \det(B_{i,n}) + t^n \prod_{n-1 \geq i > j} (a_i - a_j)
\end{aligned}
$$

where $\det(B_{n+1,n+1}) = \prod_{n-1\geq i>j} (a_i - a_j)$ is the Vandermonde formula for $n-1$. Moreover $f(a_0) = \cdots = f(a_{n-1}) = 0$ since in each case we are computing the determinant of a matrix that has two identical columns. Hence $f(t)$ is a polynomial of degree n that has the n real numbers $a_0, \ldots, a_{n-1}$ as roots, and the coefficient of t^n is $\prod_{n-1\geq i>j} (a_i - a_j)$, so

$$
f(t) = \left(\prod_{n-1 \geq i > j} (a_i - a_j) \right) (t - a_0) \cdots (t - a_{n-1}).
$$

Thus

$$
\det(A) = f(a_n) = \left(\prod_{n-1 \geq i > j} (a_i - a_j) \right) (a_n - a_0) \cdots (a_n - a_{n-1}) = \prod_{n \geq i > j} (a_i - a_j)
$$

as desired.

Problem 2($\star$).

Find

$$
\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix}
$$

using Vandermonde's formula and using the usual definition of determinant.

Solution: We have $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $a_4 = 5$, and

$$
\det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \\ 1 & 16 & 81 & 256 & 625 \end{bmatrix} = \prod_{4 \geq i > j} (a_i - a_j) = 288.
$$

Problem 3.

For n distinct scalars $a_1, \ldots, a_n$, find

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix}.$$

Solution: Factoring out one a_i from the i-th column, consecutively, we find that

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix} = a_1 \det \begin{bmatrix} 1 & a_2 & \cdots & a_n \\ a_1 & a_2^2 & \cdots & a_n^2 \\ a_1^2 & a_2^3 & \cdots & a_n^3 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^n & \cdots & a_n^n \end{bmatrix} =$$

$$= a_1 a_2 \det \begin{bmatrix} 1 & 1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n^2 \\ a_1^2 & a_2^2 & \cdots & a_n^3 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^n \end{bmatrix} = \cdots =$$

$$= (a_1 \cdots a_n) \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix} =$$

$$= \left(\prod_{i=1}^{n} a_i\right) \left(\prod_{n \geq i > j} (a_i - a_j)\right)$$

where we have used the Vandermonde formula for the last determinant.

Problem 4.

In his groundbreaking text Ars Magna, the Italian mathematician Gerolamo Cardano explains how to solve cubic equations. In Chapter XI, he considers the following example: $x^3 + 6x = 20$.

(a) Explain why this equation has exactly one (real) solution. Here, this solution is easy to find by inspection. The point of this exercise is to show a systematic way to find it.

(b) Cardano explains his method as follows (we are using modern notation for the variables): "I take two cubes v^3 and u^3 whose difference shall be 20, so that the product vu shall be 2, that is, a third of the coefficient of the unknown x. Then, I say that $v-u$ is the value of the unknown x". Show that if v and u are chosen as stated by Cardano, then $x = v - u$ is indeed the solution of the equation $x^3 + 6x = 20$.

(c) Solve the system

$$\begin{aligned} v^3 - u^3 &= 20 \\ vu &= 2 \end{aligned}$$

to find u and v.

(d) Consider the equation $x^3 + px = q$, where p is positive. Using your previous work as a guide, show that the unique solution of this equation is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

Check that this solution can also be written as

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}.$$

What can go wrong when p is negative?

(e) Consider an arbitrary cubic equation $x^3 + ax^2 + bx + c = 0$. Show that the substitution $x = t - (a/3)$ allows you to write this equation as $t^3 + pt = q$.

Solution:

(a) Consider the polynomial $f(x) = x^3 + 6x - 20$, so we have $x^3 + 6x = 20$ if and only if $f(x) = 0$. Now $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$, so f has at least one root. Since $f'(x) = 3x^2 + 6$ is always positive, f is always increasing, so f has exactly one root.

(b) Let u and v such that $v^3 - u^3 = 20$ and $uv = 2$, and set $x = v - u$. Then

$$\begin{aligned} x^3 + 6x &= (v-u)^3 + 6(v-u) = v^3 - 3v^2u + 3vu^2 - u^3 + 6(v-u) \\ &= v^3 - u^3 - 3vu(v-u) + 6(v-u) = 20 - 6(v-u) + 6(v-u) = 20. \end{aligned}$$

(c) From the second equation we have $u = 2/v$, and substituting into the first equation gives $(v^3)^2 - 20v^3 - 8 = 0$. This has solutions $v^3 = 10 \pm 6\sqrt{3}$ so $v = \sqrt[3]{10 \pm 6\sqrt{3}}$. Now $u^3 = v^3 - 20 = -10 \pm 6\sqrt{3}$ so $u = \sqrt[3]{-10 \pm 6\sqrt{3}}$.

(d) Let $v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$ and $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$. Then $v^3 - u^3 = q$ and $vu = p/3$, so since $x = v - u$ we have

$$\begin{aligned} x^3 + px &= v^3 - 3v^2u + 3vu^2 - u^3 + p(v-u) \\ &= v^3 - u^3 - 3vu(v-u) + p(v-u) = q - p(v-u) + p(v-u) = q. \end{aligned}$$

If p is negative then $(q/2)+(p/3)^3$ may be negative, and the equation $x^3+px = q$ may have more than one solution.

(e) Let $x = t - (a/3)$ and assume $x^3 + ax^2 + bx + c = 0$, then

$$\begin{aligned} 0 = x^3 + ax^2 + bx + c &= (t - (a/3))^3 + a(t - (a/3))^2 + b(t - (a/3)) + c \\ &= t^3 + (b - (a^2/3))t + (c + (2a^3/27) - (ab/3)) \end{aligned}$$

and thus $t^3 + (b - (a^2/3))t = (ab/3) - c - (2a^3/27)$. Setting $p = b - (a^2/3)$ and $q = (ab/3) - c - (2a^3/27)$ we have $t^3 + pt = q$.

Problem 5.

Consider an $n \times n$ matrix A. A subspace V of $\mathbb{R}^n$ is said to be A-invariant if $A\vec{v}$ is in V for all $\vec{v}$ in V. Describe all the one-dimensional A-invariant subspaces of $\mathbb{R}^n$ in terms of the eigenvectors of A.

Solution: Let V be a one dimensional A-invariant subspace, let $\vec{v} \in V$. Since V is one dimensional, we have $V = \text{span}(\vec{v})$. Since $\vec{v} \in V$ then $A\vec{v} \in V$ so $A\vec{v} = \lambda\vec{v}$, meaning that λ is an eigenvalue of A with associated eigenvector $\vec{v}$. Thus every one dimensional A-invariant subspace V is the span of an eigenvector of A.

We now prove the converse, namely, let $\vec{v}$ be an eigenvector of A of eigenvalue λ. Then $V = \text{span}(\vec{v})$ is one dimensional. Moreover V is an A-invariant subspace since any $\vec{w} \in V$ can be written as $c\vec{v}$ a scalar multiple of $\vec{v}$, and thus $A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v}$ so $A\vec{w} \in V$ for all $\vec{w} \in V$. Thus the span of an eigenvector of A is a one dimensional A-invariant subspace.

Problem 6($\star$).

Consider an arbitrary $n\times n$ matrix A. What is the relationship between the characteristic polynomials of A and A^T ? What does your answer tell you about the eigenvalues of A and A^T?

Solution: They coincide. To see this, we use that the determinant of a matrix and the determinant of its transpose coincide, so

$$\begin{aligned} f_A(\lambda) &= \det(A - \lambda I_n) = \det(A - \lambda I_n)^T \\ &= \det(A^T - \lambda I_n^T) = \det(A^T - \lambda I_n) = f_{A^T}(\lambda). \end{aligned}$$

Thus A and A^T have the same eigenvalues, with the same algebraic multiplicities.

Problem 7.

Suppose matrix A is similar to B. What is the relationship between the characteristic polynomials of A and B? What does your answer tell you about the eigenvalues of A and B?

Solution: They coincide. To see this, we use that the determinant is multiplicative and that the determinant of an invertible matrix is the multiplicative inverse of the determinant of its inverse. Suppose that $B = S^{-1}AS$ for some invertible matrix S, then

$$\begin{aligned} f_B(\lambda) &= \det(B - \lambda I_n) = \det(S^{-1}AS - \lambda I_n) = \det(S^{-1}AS - \lambda S^{-1} I_n S) \\ &= \det(S^{-1}(A - \lambda I_n)S) = \det(S^{-1}) \det(A - \lambda I_n) \det(S) \\ &= \det(S^{-1}) \det(S) \det(A - \lambda I_n) = \det(A - \lambda I_n) = f_A(\lambda). \end{aligned}$$

Thus A and B have the same eigenvalues, with the same algebraic multiplicities.