Math 33A
Linear Algebra and Applications

Discussion for July 25-29, 2022

Problem 1.

Consider the $n \times n$ matrix

$$J_n(k) = \begin{bmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ 0 & k & 1 & \cdots & 0 & 0 \\ 0 & 0 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & 0 & k \end{bmatrix}$$

(with all k's on the diagonal and 1's directly above), where k is an arbitrary constant. Find the eigenvalue(s) of $J_n(k)$, and determine their algebraic and geometric multiplicities.

Problem 2($\star$).

Are the following matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 3.

Consider a nonzero 3×3 matrix A such that $A^2 = 0$.

(a) Show that the image of A is a subspace of the kernel of A.

(b) Find the dimensions of the image and kernel of A.

(c) Pick a nonzero vector v_1 in the image of A, and write $\vec{v_1} = A\vec{v_2}$ for some $\vec{v_2}$ in $\mathbb{R}^3$. Let $\vec{v_3}$ be a vector in the kernel of A that fails to be a scalar multiple of $\vec{v_1}$. Show that $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis of $\mathbb{R}^3$.

(d) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to basis $\mathfrak{B}$.

Problem 4.

If A and B are two nonzero 3×3 matrices such that $A^2 = B^2 = 0$, is A necessarily similar to B?

Problem 5.

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix},$$

find an invertible matrix S such that

$$S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 6.

Consider an $n \times n$ matrix A such that $A^2 = 0$, with $\text{rank}(A) = r$ (above we have seen the case $n = 3$ and $r = 1$). Show that A is similar to the block matrix

$$B = \begin{bmatrix} J & 0 & \cdots & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & J & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Matrix B has r blocks of the form J along the diagonal, with all other entries being 0. To show this, proceed as in the case above: Pick a basis $\vec{v_1}, \ldots, \vec{v_r}$ of the image of A, write $\vec{v_i} = A\vec{w_i}$ for $i = 1, \ldots, r$, and expand $\vec{v_1}, \ldots, \vec{v_r}$ to a basis $\vec{v_1}, \ldots, \vec{v_r}, \vec{u_1}, \ldots, \vec{u_m}$ of the kernel of A. Show that $\vec{v_1}, \vec{w_1}, \vec{v_2}, \vec{w_2}, \ldots, \vec{v_r}, \vec{w_r}, \vec{u_1}, \ldots, \vec{u_m}$ is a basis of $\mathbb{R}^n$, and show that B is the matrix of $T(\vec{x}) = A\vec{x}$ with respect to this basis.

Problem 7($\star$).

Consider an $n \times m$ matrix A with $\text{rank}(A) = m$, and a singular value decomposition $A = U\Sigma V^T$. Show that the least-squares solution of a linear system $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \cdots + \frac{\vec{b} \cdot \vec{u_m}}{\sigma_m}\vec{v_m}.$$

Problem 8.

Consider the 4×2 matrix

$$A = \frac{1}{10}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Find the least-squares solution of the linear system

$$A\vec{x} = \vec{b} \quad \text{where} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Problem 9.

(a) Explain how any square matrix A can be written as $A = QS$, where Q is orthogonal and S is symmetric positive semidefinite. This is called the polar decomposition of A.

(b) Is it possible to write $A = S_1Q_1$, where Q_1 is orthogonal and S_1 is symmetric positive semidefinite?

Problem 10.

Find a polar decomposition $A = QS$ for

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$

Draw a sketch showing $S(C)$ and $A(C) = Q(S(C))$, where C is the unit circle centered at the origin.

Problem 11.

Show that a singular value decomposition $A = U\Sigma V^T$ can be written as

$$A = \sigma_1 \vec{u_1} \vec{v_1}^T + \cdots + \sigma_r \vec{u_r} \vec{v_r}^T.$$

Problem 12.

Find a decomposition $A = \sigma_1 \vec{u_1} \vec{v_1}^T + \sigma_2 \vec{u_2} \vec{v_2}^T$ for

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$