Math 33A
Linear Algebra and Applications

Discussion for July 25-29, 2022

Problem 1.

Consider the $n \times n$ matrix

$$J_n(k) = \begin{bmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ 0 & k & 1 & \cdots & 0 & 0 \\ 0 & 0 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & 0 & k \end{bmatrix}$$

(with all k's on the diagonal and 1's directly above), where k is an arbitrary constant. Find the eigenvalue(s) of $J_n(k)$, and determine their algebraic and geometric multiplicities.

Solution: Since $J_n(k)$ is upper triangular, its eigenvalues are its diagonal entries, so it has k as its single eigenvalue, with algebraic multiplicity n. Since

$$E_k = \ker(J_n(k) - kI_n) = \ker\left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\right) = \operatorname{span}(\vec{e_1})$$

then $\dim(E_k) = 1$, so the geometric multiplicity of k is 1.

Problem 2($\star$).

Are the following matrices similar?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution: No, since $A^2 = 0$ but $B^2 \neq 0$. If A was similar to B, then we would have an invertible matrix S satisfying $B = S^{-1}AS$, and thus $0 \neq B^2 = S^{-1}A^2S = 0$ gives a contradiction.

Problem 3.

Consider a nonzero 3×3 matrix A such that $A^2 = 0$.

(a) Show that the image of A is a subspace of the kernel of A.

(b) Find the dimensions of the image and kernel of A.

(c) Pick a nonzero vector v_1 in the image of A, and write $\vec{v_1} = A\vec{v_2}$ for some $\vec{v_2}$ in $\mathbb{R}^3$. Let $\vec{v_3}$ be a vector in the kernel of A that fails to be a scalar multiple of $\vec{v_1}$. Show that $\mathfrak{B} = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is a basis of $\mathbb{R}^3$.

(d) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$ with respect to basis $\mathfrak{B}$.

Solution:

(a) Let $\vec{v} \in \operatorname{im}(A)$, so that there is a vector $\vec{w} \in \mathbb{R}^3$ with $\vec{v} = A\vec{w}$. Now $A\vec{v} = A(A\vec{w}) = A^2\vec{w} = 0\vec{w} = \vec{0}$, so $\vec{v} \in \ker(A)$. Thus $\operatorname{im}(A) \subset \ker(A)$.

(b) By the above, $\dim(\operatorname{im}(A)) \leq \dim(\ker(A))$. Since A is non zero, then we have at least one non zero vector in the image of A, so $\dim(\operatorname{im}(A)) \geq 1$. By the rank nullity Theorem we have $\dim(\operatorname{im}(A))+\dim(\ker(A)) = 3$. Thus since the dimensions are integers, the only possibility is $\dim(\operatorname{im}(A)) = 1$ and $\dim(\ker(A)) = 2$.

(c) We have three non zero vectors, so to prove that they are a basis of $\mathbb{R}^3$ it is enough to prove that their are linearly independent. Suppose we have a relation $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$ for some real scalars c_1, c_2, c_3. Applying A to both terms of the equality we obtain $\vec{0} = c_2A\vec{v_2} = c_2\vec{v_1}$ so $c_2 = 0$, using that c_2, c_3 are in $\ker(A)$. Thus we have $c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$. Since $\vec{v_2}$ and $\vec{v_3}$ are linearly independent by construction, we have $c_2 = c_3 = 0$. Hence $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent.

(d) We have

$$B = \big[[A(\vec{v_1})]_{\mathfrak{B}} \quad [A(\vec{v_2})]_{\mathfrak{B}} \quad [A(\vec{v_3})]_{\mathfrak{B}}\big] = \big[[\vec{0}]_{\mathfrak{B}} \quad [\vec{v_1}]_{\mathfrak{B}} \quad [\vec{0}]_{\mathfrak{B}}\big] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 4.

If A and B are two nonzero 3×3 matrices such that $A^2 = B^2 = 0$, is A necessarily similar to B?

Solution: Yes. Using the previous problem, we can find a basis $\mathfrak{A}$ of $\mathbb{R}^3$ such that A is similar to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and we can also find a basis $\mathfrak{B}$ of $\mathbb{R}^3$ such that B is similar to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, since A and B are both similar to the same matrix, they are similar to each other.

Problem 5.

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix},$$

find an invertible matrix S such that

$$S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solution: Note that $A^2 = 0$, so we can use the method given above. We know that the vector $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the image of A, since it is the first column of A. Thus the way we obtain it is multiplying A by the first vector of the standard basis, namely $\vec{v_1} = A\vec{e_1}$, so we set $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. For our last element of the basis, we need a vector in the kernel of A that is not a scalar multiple of $\vec{v_1}$. Since

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we have the two relations $\vec{v_2} = -2\vec{v_1}$ and $\vec{v_3} = \vec{v_1}$, giving the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ in the kernel of A. Neither of them is a scalar multiple of $\vec{v_1}$, so we can set $\vec{v_3} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Now

$$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis of R^3 such that the linear transformation associated to A in $\mathfrak{B}$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

In particular

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as desired.

Problem 6.

Consider an $n \times n$ matrix A such that $A^2 = 0$, with $\operatorname{rank}(A) = r$ (above we have seen the case $n = 3$ and $r = 1$). Show that A is similar to the block matrix

$$B = \begin{bmatrix} J & 0 & \cdots & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & J & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \quad \text{where} \quad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Matrix B has r blocks of the form J along the diagonal, with all other entries being 0. To show this, proceed as in the case above: Pick a basis $\vec{v_1}, \ldots, \vec{v_r}$ of the image of A, write $\vec{v_i} = A\vec{w_i}$ for $i = 1, \ldots, r$, and expand $\vec{v_1}, \ldots, \vec{v_r}$ to a basis $\vec{v_1}, \ldots, \vec{v_r}, \vec{u_1}, \ldots, \vec{u_m}$ of the kernel of A. Show that $\vec{v_1}, \vec{w_1}, \vec{v_2}, \vec{w_2}, \ldots, \vec{v_r}, \vec{w_r}, \vec{u_1}, \ldots, \vec{u_m}$ is a basis of $\mathbb{R}^n$, and show that B is the matrix of $T(\vec{x}) = A\vec{x}$ with respect to this basis.

Solution: To show that $\vec{v_1}, \vec{w_1}, \vec{v_2}, \vec{w_2}, \ldots, \vec{v_r}, \vec{w_r}, \vec{u_1}, \ldots, \vec{u_m}$ is a basis of $\mathbb{R}^n$ it is enough to prove that there are n of them and that they are linearly independent.

Since $\vec{v_1}, \ldots, \vec{v_r}$ form a basis of the image of A we have $\dim(\operatorname{im}(A)) = r$. Since $\vec{v_1}, \ldots, \vec{v_r}, \vec{u_1}, \ldots, \vec{u_m}$ is a basis of the kernel of A then $\dim(\ker(A)) = r + m$. Thus by the rank nullity Theorem we have $n = \dim(\operatorname{im}(A)) + \dim(\ker(A)) = r + r + m = 2r + m$ so indeed there are n vectors in the list $\vec{v_1}, \vec{w_1}, \vec{v_2}, \vec{w_2}, \ldots, \vec{v_r}, \vec{w_r}, \vec{u_1}, \ldots, \vec{u_m}$.

To see that they are linearly independent, suppose we have a linear combination $a_1\vec{v_1} + b_1\vec{w_1} + \cdots + a_r\vec{v_r} + b_r\vec{w_r} + c_1\vec{u_1} + \cdots + c_m\vec{u_m} = \vec{0}$. Applying A to both sides of the equality we obtain $b_1\vec{v_1} + \cdots + b_r\vec{v_r} = \vec{0}$ so $b_1 = \cdots = b_r = 0$ since $\vec{v_1}, \ldots, \vec{v_r}$ are linearly independent. We then have $a_1\vec{v_1} + \cdots + a_r\vec{v_r} + c_1\vec{u_1} + \cdots + c_m\vec{u_m} = \vec{0}$, so $a_1 = \cdots = a_r = c_1 = \cdots = c_m = 0$ since $\vec{v_1}, \ldots, \vec{v_r}, \vec{u_1}, \ldots, \vec{u_m}$ are linearly independent.

What remains is to show that the matrix B is similar to A with respect to this change of basis. Note that for each $i = 1, \ldots, r$ the pair $\vec{v_i}, \vec{w_i}$ will contribute with a block

$$J = \big[[A(\vec{v_i})]_{\{\vec{v_i}, \vec{w_i}\}} \quad [A(\vec{w_i})]_{\{\vec{v_i}, \vec{w_i}\}}\big] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{i,i} & b_{i,i+1} \\ b_{i+1,i} & b_{i+1,i+1} \end{bmatrix}$$

to the matrix B, these blocks having their diagonal coincide with the diagonal of B. Moreover, since $A\vec{u_j} = \vec{0}$ for all $j = 1, \ldots, m$, all the other entries of the matrix B are zero.

Problem 7($\star$).

Consider an $n \times m$ matrix A with $\operatorname{rank}(A) = m$, and a singular value decomposition

$A = U\Sigma V^T$. Show that the least-squares solution of a linear system $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \cdots + \frac{\vec{b} \cdot \vec{u_m}}{\sigma_m}\vec{v_m}.$$

Solution: For some vector $\vec{x}^*$ to be a least-squares solution it just needs to satisfy $A\vec{x}^* = \text{proj}_{\text{im}(A)}(\vec{b})$. Since $\vec{u_1}, \ldots, \vec{u_n}$ are an orthonormal basis of $\mathbb{R}^n$ then

$$\begin{aligned} A\vec{x}^* &= A\left(\frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \cdots + \frac{\vec{b} \cdot \vec{u_m}}{\sigma_m}\vec{v_m}\right) = \vec{b} \cdot \vec{u_1}\frac{A\vec{v_1}}{\sigma_1} + \cdots + \vec{b} \cdot \vec{u_m}\frac{A\vec{v_m}}{\sigma_m} \\ &= (\vec{b} \cdot \vec{u_1})\vec{u_1} + \cdots + (\vec{b} \cdot \vec{u_m})\vec{u_m} = \text{proj}_{\text{im}(A)}(\vec{b}) \end{aligned}$$

because $\vec{u_1}, \ldots, \vec{u_m}$ is an orthonormal basis of $\text{im}(A)$. Thus $\vec{x}^*$ is a least-squares solution, as desired.

Problem 8.

Consider the 4×2 matrix

$$A = \frac{1}{10}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}.$$

Find the least-squares solution of the linear system

$$A\vec{x} = \vec{b} \quad \text{where} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution: We can read off the decomposition of A the following

$$\vec{u_1} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u_2} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \vec{v_1} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}, \vec{v_2} = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}, \sigma_1 = 2, \sigma_2 = 1$$

so by the above we find

$$\vec{x}^* = \frac{\vec{b} \cdot \vec{u_1}}{\sigma_1}\vec{v_1} + \frac{\vec{b} \cdot \vec{u_2}}{\sigma_2}\vec{v_2} = \begin{bmatrix} -1/10 \\ -16/5 \end{bmatrix}.$$

Problem 9.

(a) Explain how any square matrix A can be written as $A = QS$, where Q is orthogonal and S is symmetric positive semidefinite. This is called the polar decomposition of A.

(b) Is it possible to write $A = S_1 Q_1$, where Q_1 is orthogonal and S_1 is symmetric positive semidefinite?

Solution:

(a) Let $A = U\Sigma V^T$ be the singular value decomposition of A. Set $Q = UV^T$ and $S = V\Sigma V^T$, we can rewrite

$$A = U\Sigma V^T = UV^T V\Sigma V^T = QS$$

where Q is orthogonal because it is the product of orthogonal matrices, and S is symmetric since

$$S^T = (V\Sigma V^T)^T = (V^T)^T \Sigma^T V^T = V\Sigma V^T$$

because Σ only has non zero entries in its diagonal. Moreover, since S is similar to Σ then they have the same eigenvalues, and the eigenvalues of Σ are its diagonal entries, which are all positive or zero. Thus S is positive semidefinite.

(b) Yes. Set $S_1 = U\Sigma U^T$ and $Q_1 = UV^T$ and rewrite

$$A = U\Sigma V^T = U\Sigma U^T UV^T = S_1 Q_1$$

where, as we just saw, Q_1 and S_1 are orthogonal and symmetric positive semidefinite.

Problem 10.

Find a polar decomposition $A = QS$ for

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$

Draw a sketch showing $S(C)$ and $A(C) = Q(S(C))$, where C is the unit circle centered at the origin.

Solution: We compute its singular value decomposition and obtain

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\right)\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right) = U\Sigma V^T$$

so

$$Q = \left(\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix},$$
$$S = \left(\frac{1}{\sqrt{5}}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}\right)\begin{bmatrix}10 & 0\\ 0 & 5\end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\right) = \begin{bmatrix}9 & -2\\ -2 & 6\end{bmatrix},$$

and

$$A = \left(\frac{1}{5}\begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}\right)\begin{bmatrix}9 & -2\\ -2 & 6\end{bmatrix}.$$

Problem 11.

Show that a singular value decomposition $A = U\Sigma V^T$ can be written as

$$A = \sigma_1 \vec{u_1}\vec{v_1}^T + \cdots + \sigma_r \vec{u_r}\vec{v_r}^T.$$

Solution: We can rewrite the singular value decomposition of A as

$$A = U\Sigma V^T = \begin{bmatrix}| & & |\\ \vec{u_1} & \cdots & \vec{u_n}\\ | & & |\end{bmatrix}\begin{bmatrix}\sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0\end{bmatrix}\begin{bmatrix}— & \vec{v_1} & —\\ & \vdots & \\ — & \vec{v_m} & —\end{bmatrix}$$
$$= \begin{bmatrix}| & & |\\ \vec{u_1} & \cdots & \vec{u_n}\\ | & & |\end{bmatrix}\begin{bmatrix}— & \sigma_1\vec{v_1} & —\\ & \vdots & \\ — & \sigma_r\vec{v_r} & —\\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}$$
$$= \sigma_1\begin{bmatrix}|\\ \vec{u_1}\\ |\end{bmatrix}\begin{bmatrix}— & \vec{v_1} & —\end{bmatrix} + \cdots + \sigma_1\begin{bmatrix}|\\ \vec{u_1}\\ |\end{bmatrix}\begin{bmatrix}— & \vec{v_1} & —\end{bmatrix} = \sigma_1\vec{u_1}\vec{v_1}^T + \cdots + \sigma_r\vec{u_r}\vec{v_r}^T$$

giving the desired decomposition.

Problem 12.

Find a decomposition $A = \sigma_1\vec{u_1}\vec{v_1}^T + \sigma_2\vec{u_2}\vec{v_2}^T$ for

$$A = \begin{bmatrix}6 & 2\\ -7 & 6\end{bmatrix}.$$

Solution: We compute its singular value decomposition and obtain

$$A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix} = \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \right) \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right) = U\Sigma V^T.$$

We can read off this decomposition the following

$$\vec{u_1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \vec{u_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \vec{v_1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ \vec{v_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \sigma_1 = 10, \ \sigma_2 = 5$$

so

$$\begin{aligned} A &= 10 \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \end{bmatrix} \right) + 10 \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 8 & -2 \\ -8 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}. \end{aligned}$$