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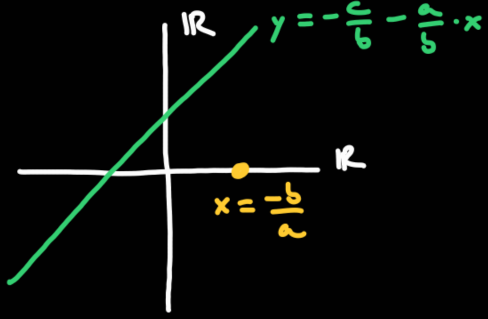
1. Introduction to Linear Algebra.

Linear algebra is the study of linear equations and linear transformations.

$$a \cdot x + b = 0 \rightsquigarrow x = \frac{-b}{a}$$

$$a \cdot x + b \cdot y + c = 0 \rightsquigarrow y = \frac{-c}{b} - \frac{a}{b} \cdot x$$



variables

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b = 0$$

coefficients constant term

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + b_1 = 0 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n + b_2 = 0 \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n + b_n = 0 \end{cases}$$

equation/row

$a_{ij} \, x_j$

variable/column

A system of linear equations has either one solution, no solutions, or infinitely many solutions.

Matrix: n rows m columns, we say that it has size $n \times m$.

a_{ij}

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Two matrices are equal when the entries are equal.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\neq$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Special families of matrices:

(i) Square matrices.

(ii) Diagonal $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

(iii) Zero matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

<u>Vector:</u> matrices with one column. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$ $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \text{components}$$

Fix n a natural number. The set of all column vectors with n components is denoted by $\mathbb{R}^n$. We say that $\mathbb{R}^n$ is a <u>vector space</u>.

(x,y) $(1,2)$

$\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n \end{cases}$$

n equations m variables

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ \underbrace{a_{n1}}_{} & a_{n2} & \cdots & a_{nm} & \underbrace{b_n}_{\text{constant terms}} \end{array}\right]$$

coefficients

Augmented matrix.

Gauss-Jordan:

(i) Divide a row by a non-zero real number.

(ii) Subtract a multiple of a row to another row.

(iii) Swap two rows.

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_3 - 7R_1]{R_2 - 4R_1} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow[R_2 \cdot \frac{1}{-3}]{} \begin{bmatrix} 1 & 2 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow[R_3 + 6R_2]{R_1 - 2R_2}$$

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The simplified matrix is called the reduced-row-echelon form. rref

(i) If a row has non-zero entries, the first non-zero entry is a 1.

leading ones.

(ii) If a column contains a leading 1, then all the other entries in the column are zero.

(iii) If a row contains a leading 1, each row above it contains a leading 1 further to the left.

System of equations:

consistent if it has at least one solution.

→ one if all variables are leading

→ infinitely many if we have a free variable.

inconsistent if it has no solutions. ⟷ $[0 \cdots 0 \mid 1]$ $\quad 0=1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \quad (*) \\ 4x_1 + 5x_2 + 6x_3 = 0 \\ 7x_1 + 8x_2 + 9x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \\ 0 = 0 \end{cases} \qquad \begin{aligned} x_1 &= t \\ x_2 &= -2t \\ x_3 &= t \end{aligned}$$

$(*)$ $t + 2\cdot(-2t) + 3t = t - 4t + 3t = 0.$

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix} \quad t \text{ real number.}$$