Review:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned} \rightsquigarrow \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{array}\right]$$

$$\underbrace{\qquad\qquad\qquad}_{\text{coefficients}} \quad \underbrace{\quad}_{\text{constant terms}}$$

$$\overset{\text{row } n}{\underset{\text{column } m}{a_{ij}}}$$

Rank: A matrix, the <u>rank</u> of A is the number of leading ones in rref(A).

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{bmatrix} \overset{a_{11}}{1} & 0 & -1 \\ 0 & \underset{a_{22}}{1} & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{rank } 2.$$

n equations $\rightsquigarrow$ n rows
m variables $\qquad$ m columns

$\text{rank}(A) \leq n \qquad \text{rank}(A) \leq m$

If $\text{rank}(A) = n$ the system is consistent.

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix} \end{array} \qquad \begin{aligned} x &= \boxed{} \\ y &= \boxed{} \end{aligned}$$

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{l} x \text{ fully determined} \\ y \text{ fully determined} \end{array}$$

If $\text{rank}(A) = m$ the system has at most one solution.

$$\begin{array}{cc} x & y \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{array}\right]$$

$x = \square$

$y = \square$

$0 = 0$ one solution

$0 = 1$ no solution

If rank(A) < n the system has zero solutions or infinitely many.

$$\left[\begin{array}{ccc|c} x & y & z & \\ 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{array}\right]$$

infinitely many $0 = 0$

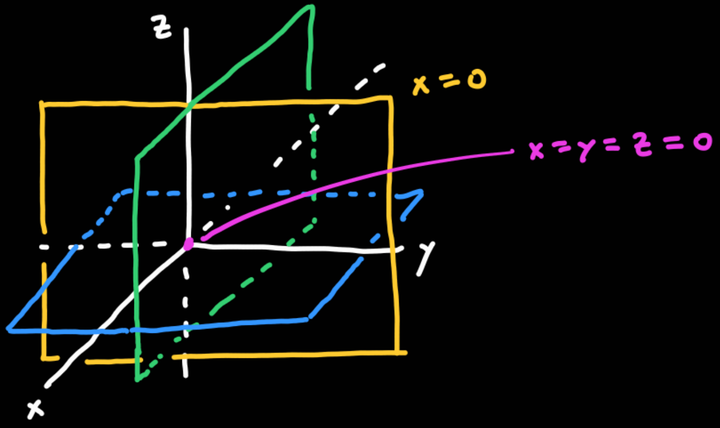
$x = \square$

$y = \square$

z real number

$0 = 1$ no solution

$$\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$$



Addition:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 12 & 14 \\ 16 & 18 \end{bmatrix}$$

Scalar multiplication:

$$8 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ 24 & 32 \\ 40 & 48 \end{bmatrix}$$

Dot product: (vectors)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = 32$$

Product of matrix with a vector:

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}7\\8\end{bmatrix}\\[4pt]\begin{bmatrix}3\\4\end{bmatrix}\cdot\begin{bmatrix}7\\8\end{bmatrix}\\[4pt]\begin{bmatrix}5\\6\end{bmatrix}\cdot\begin{bmatrix}7\\8\end{bmatrix}\end{bmatrix}=\begin{bmatrix}\underbrace{7}+\underbrace{16}\\21+32\\35+48\end{bmatrix}=\begin{bmatrix}25\\53\\83\end{bmatrix}$$

$$\underbrace{3}_{\text{rows}}\times\underbrace{2}_{\text{columns}} \qquad 3\times\underbrace{1}$$

$$\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix}=\underbrace{7\cdot\begin{bmatrix}1\\3\\5\end{bmatrix}}+\underbrace{8\cdot\begin{bmatrix}2\\4\\6\end{bmatrix}}=\begin{bmatrix}7\\21\\35\end{bmatrix}+\begin{bmatrix}16\\32\\48\end{bmatrix}=\begin{bmatrix}25\\53\\83\end{bmatrix}$$

$$\underbrace{\begin{aligned}a_{11}x_1+a_{12}x_2+a_{13}x_3&=b_1\\a_{21}x_1+a_{22}x_2+a_{23}x_3&=b_2\\a_{31}x_1+a_{32}x_2+a_{33}x_3&=b_3\end{aligned}}\ \rightsquigarrow\ \left[\begin{array}{ccc|c}a_{11}&a_{12}&a_{13}&b_1\\a_{21}&a_{22}&a_{23}&b_2\\a_{31}&a_{32}&a_{33}&b_3\end{array}\right]$$

coefficient matrix

$$\begin{bmatrix}a_{11}x_1+a_{12}x_2+a_{13}x_3\\a_{21}x_1+a_{22}x_2+a_{23}x_3\\a_{31}x_1+a_{32}x_2+a_{33}x_3\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$A \qquad \vec{x} \qquad \vec{y} \qquad k$

$$A(\vec{x}+\vec{y})=A\vec{x}+A\vec{y} \qquad\qquad A(k\vec{x})=k(A\vec{x})$$

Linear combination: $\underbrace{\mathbb{R}^3}_{\text{3-dimensional vector space}}$ $\begin{bmatrix}*\\*\\*\end{bmatrix}$

$1, 2, 3, 4 \leftarrow$ scalars

$\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}3\\2\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix} \leftarrow$ vectors

$$\vec{v}=1\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}+2\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}+3\cdot\begin{bmatrix}3\\2\\1\end{bmatrix}+4\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}$$

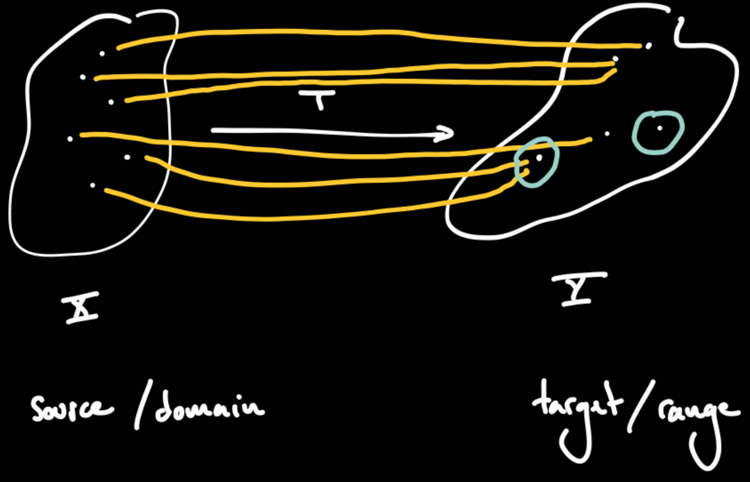
$$\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad\qquad \begin{bmatrix}a\\b\\c\end{bmatrix}=a\cdot\begin{bmatrix}1\\0\\0\end{bmatrix}+b\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}+c\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

$$\begin{bmatrix}0&2&3\\4&1&5\\6&0&0\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_6}\begin{bmatrix}6&0&0\\4&1&5\\0&2&3\end{bmatrix}\xrightarrow{\frac{1}{6}R_1}\begin{bmatrix}1&0&0\\4&1&5\\0&2&3\end{bmatrix}$$

$$\xrightarrow{R_2-4R_1}\begin{bmatrix}1&0&0\\0&1&5\\0&\boxed{2}&3\end{bmatrix}\xrightarrow{R_3-2R_2}\begin{bmatrix}1&0&0\\0&1&5\\0&0&-7\end{bmatrix}\xrightarrow{\frac{1}{-7}R_3}$$

$$\begin{bmatrix}1&0&0\\0&1&5\\0&0&1\end{bmatrix}\xrightarrow{R_2-5R_3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

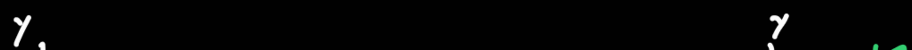
Function:

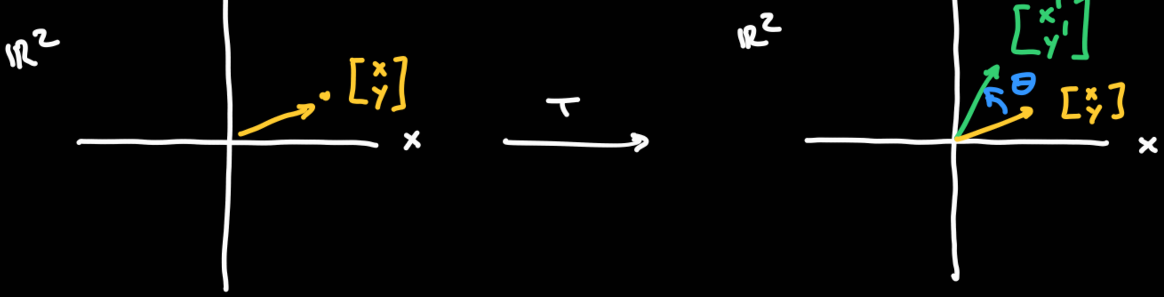


Linear transformation: $T:\mathbb{R}^m\to\mathbb{R}^n$ such that there exist an $n\times m$ matrix A with $T(\vec{x})=A\underbrace{\vec{x}}_{m \text{ entries}}$.

$$\underbrace{\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}}_{\text{does } \underline{\text{not}} \text{ give a linear transformation } \mathbb{R}^3\to\mathbb{R}^2.}$$ determines a linear function $T:\mathbb{R}^2\to\mathbb{R}^3$.

Example:



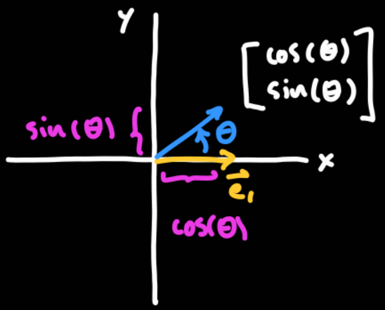


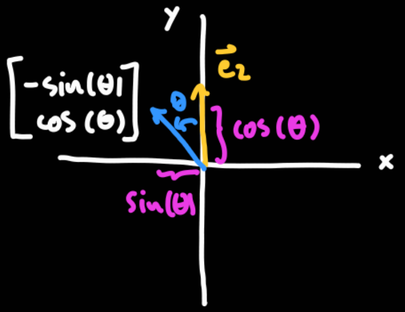
This is a linear transformation!

Express x' and y' in terms of θ, x, y.

$$x' = x\cdot\cos(\theta) - y\cdot\sin(\theta)$$
$$y' = x\cdot\sin(\theta) + y\cdot\cos(\theta)$$

$$\begin{bmatrix}x\\y\end{bmatrix} = x\cdot\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\vec{e}_1} + y\cdot\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\vec{e}_2}$$





$T: \mathbb{R}^2 \to \mathbb{R}^2$.
$\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}-\\-\end{bmatrix}$

$$\begin{bmatrix}x'\\y'\end{bmatrix} = x\cdot\begin{bmatrix}\cos(\theta)\\\sin(\theta)\end{bmatrix} + y\cdot\begin{bmatrix}-\sin(\theta)\\\cos(\theta)\end{bmatrix} = \begin{bmatrix}x\cdot\cos(\theta) - y\sin(\theta)\\x\cdot\sin(\theta) + y\cos(\theta)\end{bmatrix}$$

Find a matrix $A = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}$ such that $\begin{bmatrix}x'\\y'\end{bmatrix} = A\cdot\begin{bmatrix}x\\y\end{bmatrix}$.

$$A = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\\sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$A\cdot\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x'\\y'\end{bmatrix}.$$

⊛ $$A\cdot\begin{bmatrix}x\\y\end{bmatrix} = A\cdot\left(\begin{bmatrix}x\\0\end{bmatrix}+\begin{bmatrix}0\\y\end{bmatrix}\right) = A\begin{bmatrix}x\\0\end{bmatrix} + A\begin{bmatrix}0\\y\end{bmatrix} = A\cdot\left(x\cdot\begin{bmatrix}1\\0\end{bmatrix}\right) + A\cdot\left(y\cdot\begin{bmatrix}0\\1\end{bmatrix}\right) =$$
$$= x\cdot\left(A\begin{bmatrix}1\\0\end{bmatrix}\right) + y\cdot\left(A\cdot\begin{bmatrix}0\\1\end{bmatrix}\right).$$

⊛ Given a linear transformation T, the matrix associated to T is :

A

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_m) \end{bmatrix} = A \qquad T(\vec{x}) = A\vec{x}$$

$$T: \mathbb{R}^m \to \mathbb{R}^n \qquad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

⊛ A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if and only if:

$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and

$T(k\vec{x}) = k\,T(\vec{x})$.