Recall: Linear transformations $T: \mathbb{R}^m \to \mathbb{R}^n$ are given by multiplication with a matrix: $T(\vec{x}) = A\vec{x}$. A matrix of size $n \times m$.

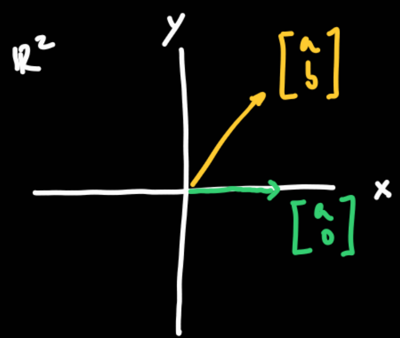
$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \qquad T(k\vec{x}) = kT(\vec{x})$$

Examples:

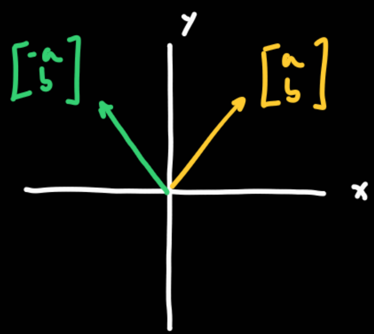
1. $$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\text{length } 2} = \underbrace{\begin{bmatrix} 10 \\ 0 \end{bmatrix}}_{\text{length } 10}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightsquigarrow \text{length } \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$$

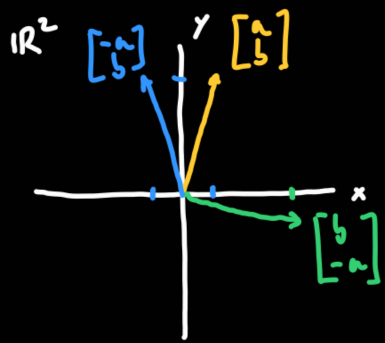
2. $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$



3. $$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$$



4. $$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix}$$

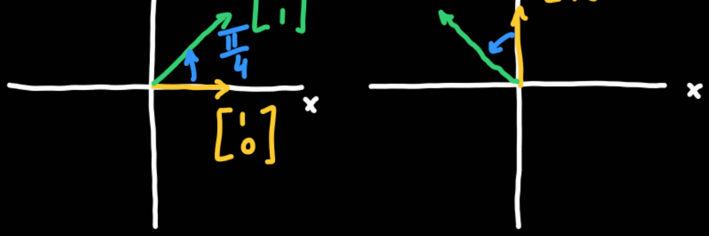


$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. $$\begin{bmatrix} 1 & -1 \\ \ldots \end{bmatrix} \begin{bmatrix} 1 \\ \ldots \end{bmatrix} = \begin{bmatrix} 1 \\ \ldots \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{1} = \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\sqrt{2}}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{1} = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\sqrt{2}}$$

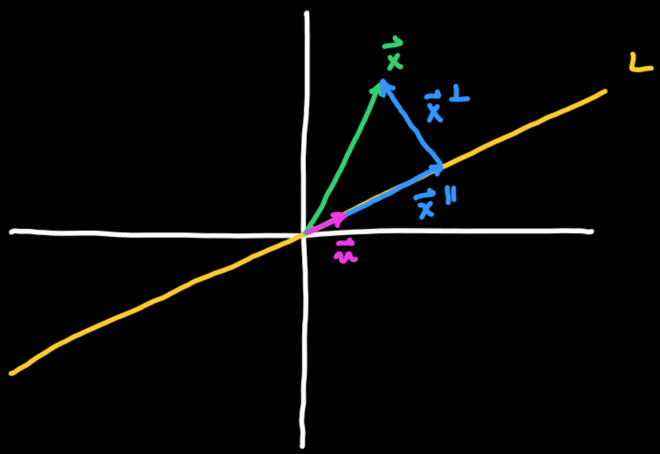


counterclockwise rotation of $\frac{\pi}{4}$.

scaling by $\sqrt{2}$.

Scaling: $\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$

Orthogonal projections:

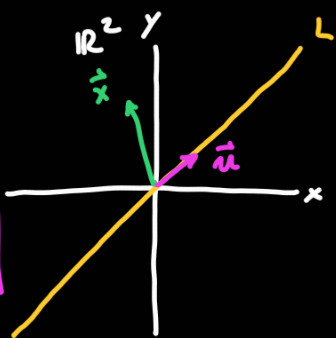


$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel} = (\vec{x}\cdot\vec{u})\,\vec{u}$$

Example: $y = x$

$y = \frac{1}{3}x \qquad \vec{u} = \frac{3}{\sqrt{10}}\begin{bmatrix} 1 \\ 1/3 \end{bmatrix}$

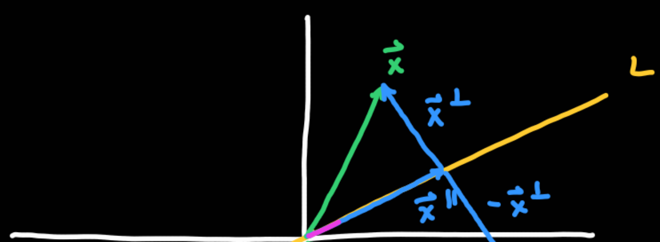


$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$

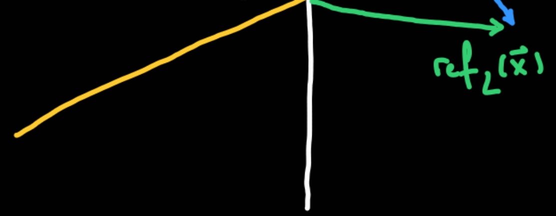
$\vec{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\,\vec{u} = \left(\begin{bmatrix} a \\ b \end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}(a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Reflections:

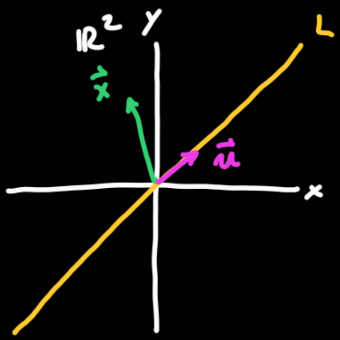


$$\begin{aligned} \text{ref}_L(\vec{x}) &= \vec{x} - \vec{x}^{\perp} - \vec{x}^{\perp} \\ &= (\vec{x}^{\parallel} + \vec{x}^{\perp}) - \vec{x}^{\perp} - \vec{x}^{\perp} \\ &= \vec{x}^{\parallel} - \vec{x}^{\perp} \end{aligned}$$



$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

Example: $y = x$



$$\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\vec{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

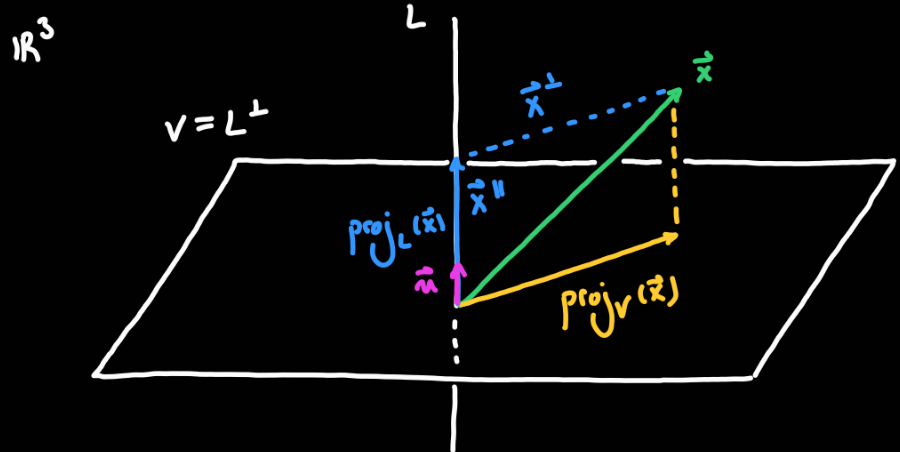
$$\vec{x}^{\parallel} = \text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u} = \left(\begin{bmatrix} a \\ b \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}(a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \begin{bmatrix} a \\ b \end{bmatrix} - \frac{(a+b)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{a}{2} - \frac{b}{2} \\ -\frac{a}{2} + \frac{b}{2} \end{bmatrix}$$

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} = \frac{(a+b)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{a}{2} - \frac{b}{2} \\ -\frac{a}{2} + \frac{b}{2} \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}$$

Three-dimensional orthogonal projections:



$$\vec{x} = \text{proj}_L(\vec{x}) + \text{proj}_V(\vec{x})$$

$$\text{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$$

$$\text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \text{proj}_V(\vec{x})$$

$\text{proj}_V(\vec{x}) = \vec{x} - (\vec{x}\cdot\vec{u})\vec{u}$ $\qquad$ $\text{ref}_V(\vec{x}) = \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x})$

Consecutive linear transformations:

$$A = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{T \text{ first}} \qquad \underbrace{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}}_{S \text{ second}} = B$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; (*) \qquad\qquad (*) \; \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{array}{ccc} T : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \vec{x} & & \vec{y} \end{array} \qquad T(\vec{x}) = \vec{y} \qquad\qquad \begin{array}{ccc} S : \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \vec{y} & & \vec{z} \end{array} \qquad S(\vec{y}) = \vec{z}$$

$$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{\;T\;} & \mathbb{R}^2 & \xrightarrow{\;S\;} & \mathbb{R}^2 \\ \vec{x} & & \vec{y} & & \vec{z} \end{array}$$

$$(S \circ T)(\vec{x}) = (BA)\vec{x}$$
$$B(A\vec{x})$$

$(*)$
$$y_1 = x_1 + 2x_2$$
$$y_2 = 3x_1 + 4x_2$$

$(*)$
$$z_1 = 5y_1 + 6y_2$$
$$z_2 = 7y_1 + 8y_2$$

$$z_1 = 5(x_1 + 2x_2) + 6(3x_1 + 4x_2) = 23x_1 + 34x_2$$
$$z_2 = 7(x_1 + 2x_2) + 8(3x_1 + 4x_2) = 31x_1 + 46x_2$$

$$\left(\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ first } T \text{ then } S$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5\cdot1 + 6\cdot3 & 5\cdot2 + 6\cdot4 \\ 7\cdot1 + 8\cdot3 & 7\cdot2 + 8\cdot4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$f, g \qquad f : \mathbb{R} \to \mathbb{R} \qquad g : \mathbb{R} \to \mathbb{R}$

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{g} \mathbb{R} \qquad (g \circ f)(x) = g(f(x))$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ if T is linear

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} \text{linear equation} \\ \text{linear equation} \end{bmatrix}$$

$$a_{11} x_1 + a_{12} x_2$$

$$a_{21} x_1 + a_{22} x_2 \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$x_1 x_2$