Recall:

$$\begin{array}{l}\text{1 row}\\ \text{1 row}\end{array}\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}\ \overset{\text{1 column}}{\underset{3\times 2}{\begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}}} = \begin{bmatrix} 7+16+27 & 10+22+36 \\ 28+40+54 & \end{bmatrix} = \underset{2\times 2}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

2 column

3 columns 3 rows

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$ is a 2×1 matrix, but $\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is <u>not</u> defined!

Note that the product of two matrices $\overset{n\times p}{A}$ and $\overset{q\times m}{B}$ is only defined when the number of columns of A coincides with the number of rows of B. If $p=q$ then AB is an $n\times m$ matrix.

$$AB = A\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_m \\ | & & | \end{bmatrix}$$

$C = AB$ has entry
$$\begin{bmatrix} - & \vec{w}_1 & - \\ & \vdots & \\ - & \vec{w}_n & - \end{bmatrix}\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = \begin{bmatrix} c_{ij} = \vec{w}_i \cdot \vec{v}_j \end{bmatrix}$$

$$\begin{bmatrix} a_{ij} \end{bmatrix}\begin{bmatrix} b_{ij} \end{bmatrix} \qquad c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

Example:

$$\overbrace{\overbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}^{T}\overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}^{S}}^{TS} = \begin{bmatrix} 1 & 3 \\ 3 & 7 \end{bmatrix}$$

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad S: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$TS: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\nparallel$

$ST: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

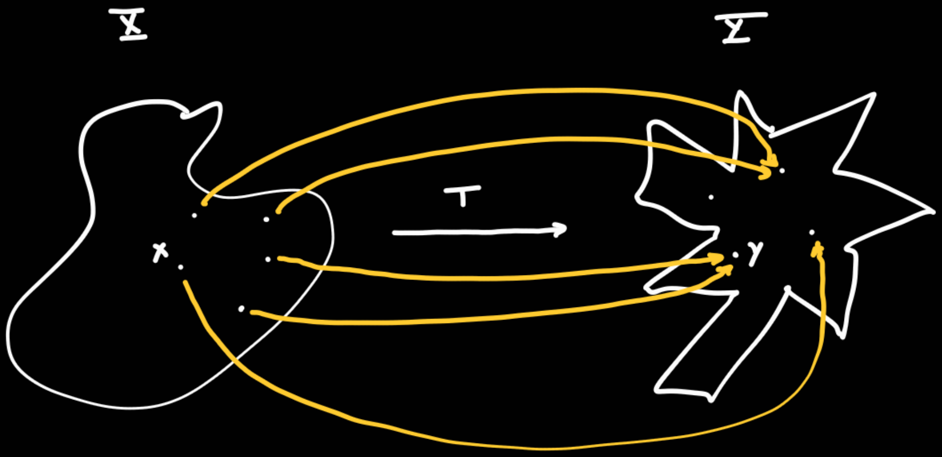
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 3 & 4 \end{bmatrix}$$

$$\underbrace{\underbrace{[\quad]}_{S}\underbrace{[\quad]}_{T}}_{ST}$$

Algebraic rules: matrices behave like real numbers, with the role of the number one is done by the matrices $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ called identity matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Functions:

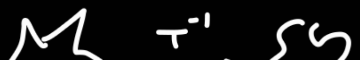


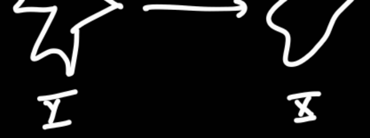
To each x in X we associate a y in Y.

A function is invertible if for each y in Y we associated a unique x in X.

The inverse of T, denoted T^{-1}, is the function that to each y in Y associates x in X when $T(x)=y$.

$$T^{-1}(y)=x \quad \text{if and only if} \quad T(x)=y.$$





$T(T^{-1}(y)) = y$ for all y in $\mathbb{Y}$. $T^{-1}(T(x)) = x$ for all x in $\mathbb{X}$.

Invertible linear transformations:

$T(\vec{x}) = A\vec{x}$, then T^{-1} will be a linear transformation!

$T^{-1}(\vec{y}) = B\vec{x}$, we denote $A^{-1} = B$.

$\vec{y} = A\vec{x}$ $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$y_1 = a_{11}x_1 + \cdots + a_{1n}x_n$
$\vdots$
$y_n = a_{n1}x_1 + \cdots + a_{nn}x_n$

$\vec{x} = A^{-1}\vec{y}$

A given
y given

$x = y$
$x = y$
$x = y$

Let A be an $n \times n$ matrix.

A is invertible if and only if $\operatorname{rank}(A) = n$.

$\operatorname{rref}(A) = I_n$.

If A is <u>not</u> square, it will <u>not</u> have an inverse.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $T(\vec{x}) = A\vec{x}$ $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{bmatrix} x_1 \end{bmatrix} \mapsto A\vec{x} = \begin{bmatrix} x_1 + x_2 \end{bmatrix}$

$$\begin{bmatrix} \\ x_2 \end{bmatrix} \mapsto \underbrace{A\vec{x}}_{\text{given } ⊛} = \begin{bmatrix} \\ x_2 \end{bmatrix}$$

$$T^{-1}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{given}} \mapsto \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We know that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$ because $\underbrace{T(\vec{x}) = \vec{y}}_{\text{given, found using } ⊛}$.

$$\overset{\text{known}}{y_1} = x_1 + x_2 \qquad x_1 = y_1 - x_2 \qquad x_1 = y_1 - y_2$$
$$y_2 = x_2 \qquad x_2 = y_2 \qquad x_2 = y_2$$

$$T^{-1}: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 - y_2 \\ y_2 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A^{-1}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2×1 $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a matrix of rank 1.

$$T: \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$[x_1] \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}[x_1] = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$
$$[x_1] \leftarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad y_2 = 1$$

1×2 $\begin{bmatrix} * & * \end{bmatrix}$

<u>Example</u>: Rotation is a linear transformation. Is it invertible?

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad \text{rotation } \circlearrowleft \text{ by } \theta$$

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

rotation ↻ by θ

rotation ↺ by $-\theta$

even function: $\cos(\theta) = \cos(-\theta)$

odd function: $\sin(-\theta) = -\sin(\theta)$

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example: Use Gauss-Jordan to compute inverses:

$$\underbrace{\begin{array}{cc|cc} x_1 & x_2 & y_1 & y_2 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}}_{A}$$

$x_1 + x_2 = y_1$

$x_2 = y_2$

$$\Big\downarrow R_1 - R_2$$

$$\left[\underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{I_n \text{ when } A \text{ invertible}} \middle| \underbrace{\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array}}_{A^{-1}}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Multiplying a matrix by its inverse gives the identity.

Suppose A, B square invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

$(ab)^{-1} = a^{-1}b^{-1}$

$\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, shift left, "B"] \arrow[rr, bend left=50, "AB"] & \mathbb{R}^n \arrow[l, shift left, "B^{-1}"] \arrow[r, shift left, "A"] & \mathbb{R}^n \arrow[l, shift left, "A^{-1}"] \arrow[ll, bend left=50, "(AB)^{-1} = B^{-1}A^{-1}"] \end{tikzcd}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If A, B are $n \times n$ matrices such that $BA = I_n$ then:

A, B are both invertible,

$A^{-1} = B$ and $B^{-1} = A$,

$AB = I_n$.