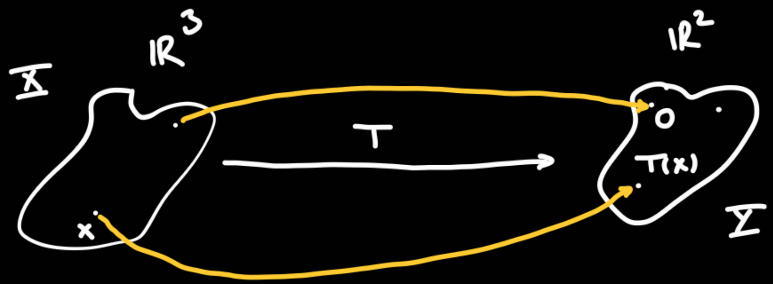
Recall: Finding inverses is just solving a system of linear equations.

Gauss-Jordan.

2. Subspaces of $\mathbb{R}^n$.



T linear.

$\ker(T) = \{x \in X \mid T(x) = 0\}$

Kernel

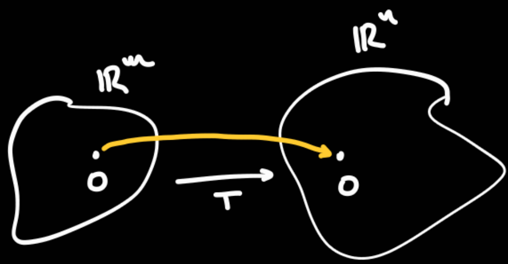
$\mathrm{im}(T) = \{T(x) \mid x \text{ in } X\} = \{y \in Y \mid \text{there is } x \text{ in } X \text{ with } T(x) = y\}$.

image

Why does 0 in $\mathbb{R}^2$ have an arrow pointing to it?

Why is 0 in $\mathrm{im}(T)$ for all T?

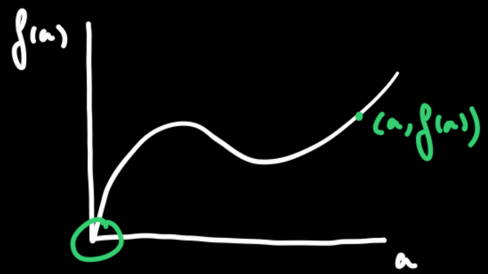
$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



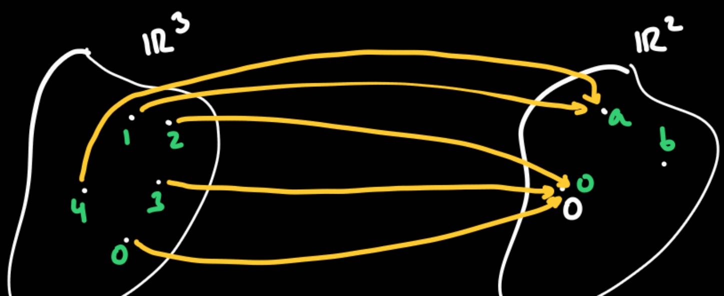
$f : \mathbb{R} \to \mathbb{R}$

$a \mapsto f(a)$

$(a, f(a))$



The graph of a linear function always goes through zero.



$\text{im}(T) = \{a, 0\}$

$\ker(T) = \{0, 2, 3\}$

Find the image of:

$$\begin{bmatrix}1&3&0\\1&4&2\end{bmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

We want $\begin{bmatrix}y_1\\y_2\end{bmatrix}$ such that there is $\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\text{we can move } x_1,\, x_2,\, x_3}$ with $\begin{bmatrix}1&3&0\\1&4&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$.

$$\begin{bmatrix}1&3&0\\1&4&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1 + 3x_2\\ x_1 + 4x_2 + 2x_3\end{bmatrix}$$

$$\begin{bmatrix}1&3&0\\1&4&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = x_1 \cdot \begin{bmatrix}1\\1\end{bmatrix} + x_2 \cdot \begin{bmatrix}3\\4\end{bmatrix} + x_3 \cdot \begin{bmatrix}0\\2\end{bmatrix} =$$

$$= \underbrace{(x_1 + 3x_2)}_{\text{any real number}} \cdot \begin{bmatrix}1\\0\end{bmatrix} + \underbrace{(x_1 + 4x_2 + 2x_3)}_{\text{any real number}} \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix} \quad a, b \text{ real numbers.}$$

$\text{im}(T) = \mathbb{R}^2$

a

b

$x_1 = a$

$x_2 = 0$

$a + 2x_3 = b \rightsquigarrow x_3 = \frac{b-a}{2}$

$$\begin{bmatrix}1&3&0\\1&4&2\end{bmatrix}\underbrace{\begin{bmatrix}a\\0\\\frac{b-a}{2}\end{bmatrix}}_{\mathbb{R}^3} = \underbrace{\begin{bmatrix}a\\b\end{bmatrix}}_{\mathbb{R}^2}$$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = (*)$$

$$= (x_1 + x_2)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (x_1 + 2x_2)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (x_1 + 3x_2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} =$$

The vector $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is <u>not</u> in $\mathrm{im}(T)$.

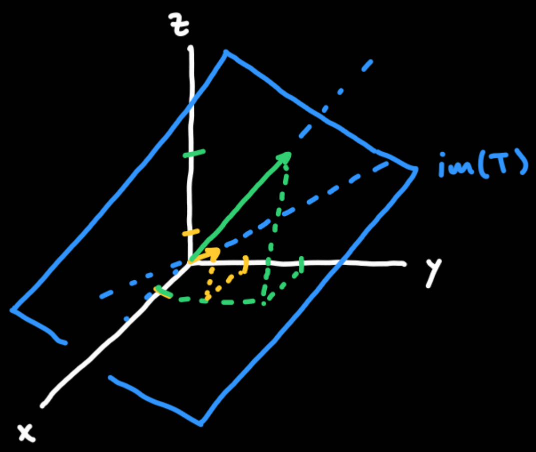
$$(*) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix} \qquad \begin{aligned} a &= x_1 + x_2 \\ b &= x_1 + 2x_2 \\ c &= x_1 + 3x_2 \end{aligned}$$

If we want $b = c = 0$ then the system $\begin{aligned} 0 &= x_1 + 2x_2 \\ 0 &= x_1 + 3x_2 \end{aligned}$

has <u>one</u> solution, $x_1 = x_2 = 0$. Then $a = 0$.

The image of a linear transformation is the span (linear combination) of its columns.

$$\mathrm{im}(T) = \left\{ x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \;\middle|\; x_1, x_2 \text{ are real numbers} \right\}$$



$$S : \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \mathrm{im}(S) = \left\{ x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;\middle|\; x_1 \text{ in } \mathbb{R} \right\}$$

$$x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} =$$

$$= x_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \cdot x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$= (x_1 + 3x_2) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\ker(S) = \{ \vec{x} \text{ in } \mathbb{R}^2 \mid S(\vec{x}) = \vec{0} \}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solve for x_1, x_2

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free variable

$x_1 + 3x_2 = 0$

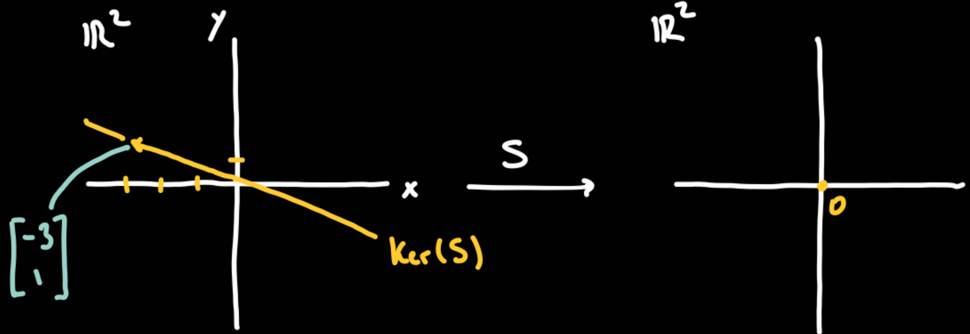
$x_2 = t$

t in $\mathbb{R}$

$x_1 = -3t$

$$\begin{bmatrix} -3t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\ker(S) = \left\{ t \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} \,\middle|\, t \text{ in } \mathbb{R} \right\}$$



$\mathbb{R}^n$ vector space.

$\mathrm{im}(T)$ and $\ker(T)$ are examples of subspaces of $\mathbb{R}^n$.

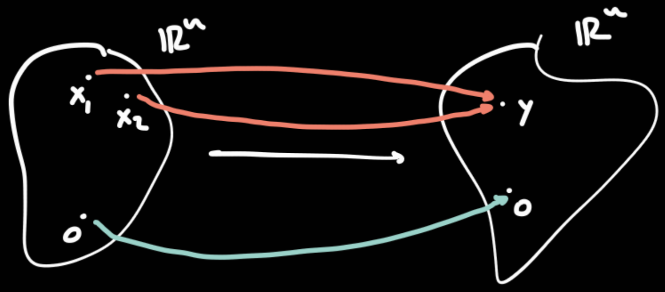
If T is invertible then $\ker(T) = \{\vec{0}\}$ and $\mathrm{im}(T) = \mathbb{R}^n$. This also goes

A

$$T: \mathbb{R}^n \to \mathbb{R}^n.$$

in the other direction: if T satisfies that $\ker(T) = \{\vec{0}\}$ and $\mathrm{im}(T) = \mathbb{R}^n$

then T is invertible.



$$T(x_1 - x_2) = T(x_1) - T(x_2) = y - y = 0$$

$x_1 - x_2$ is in $\ker(T) = \{\vec{0}\}$ so $x_1 - x_2 = 0$ so $x_1 = x_2$.

Relation between rank / kernel / image:

If $\ker(A) = \{\vec{0}\}$ then $\operatorname{rank}(A) = m$. Also $m = \operatorname{rank}(A) \leq n$.

If $m > n$ then $\ker(A) \neq \{\vec{0}\}$.

If A is a square matrix then $\ker(A) = \{\vec{0}\}$ if and only if A invertible.

A linear subspace of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ satisfying:

(i) $\vec{0}$ is in.

(ii) Closed under addition.

(iii) Closed under scalar multiplication.

Let $\vec{v}_1, \dots, \vec{v}_j$ be vectors in $\mathbb{R}^n$. The span of $\vec{v}_1, \dots, \vec{v}_j$ are the sums:

$\underbrace{a_1\vec{v}_1 + \dots + a_j\vec{v}_j}_{\text{linear combination of } \vec{v}_1, \dots, \vec{v}_j}$ for all $a_1, \dots, a_j$ in $\mathbb{R}$.

$\mathbb{R}^3$ is the span of $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$.

$$\underbrace{\begin{bmatrix}a_1\\ \vdots\\ a_j\end{bmatrix}}_{\text{fixed}} \longmapsto a_1\vec{v}_1 + \cdots + a_j\vec{v}_j$$

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \longmapsto 1\cdot\begin{bmatrix}1\\1\end{bmatrix} + 2\cdot\begin{bmatrix}1\\2\end{bmatrix} + 3\cdot\begin{bmatrix}7\\11\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 7\\ 1 & 2 & 11\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\text{im}(T) = \left\{ a_1\cdot\begin{bmatrix}\vec{v}_1\end{bmatrix} + a_2\cdot\begin{bmatrix}\vec{v}_2\end{bmatrix} + \cdots + a_j\begin{bmatrix}\vec{v}_j\end{bmatrix} \right\}$$

Examples of the type: here you have a subspace, find the matrix.

Find a matrix whose image is the vector space:

$$V = \left\{ a_1\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} + a_2\cdot\begin{bmatrix}3\\2\\1\end{bmatrix} \;\middle|\; a_1, a_2 \text{ real numbers} \right\}$$

A such that $\text{im}(A) = V$.

$$A = \begin{bmatrix}1 & 3\\ 1 & 2\\ 0 & 1\end{bmatrix} \qquad A\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1\begin{bmatrix}1\\1\\0\end{bmatrix} + x_2\cdot\begin{bmatrix}3\\2\\1\end{bmatrix}$$

V is given by
$$\begin{aligned} x + y + z &= 0\\ 3x + 2y + z &= 0 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\ 3 & 2 & 1 & 0\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\ \end{array}\right.$$

$$A = \begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \end{array}\right]$$

$$V = \left\{ t \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \;\middle|\; t \text{ real number} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$x - z = 0$$

$$y + 2z = 0$$

$$z = t \text{ real number}$$

$$\begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

L $\quad x + y = 0$

S $\quad x + y = 1$