<u>Recall:</u> subspaces of $\mathbb{R}^n$

equation $\longleftrightarrow$ span (linear combination)

$x+y+z=0$ $\qquad\qquad$ $a \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ a, b real numbers

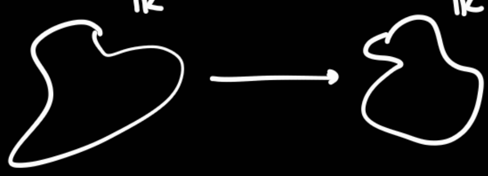
plane in 3 dimensions $\qquad\qquad$ plane in 3 dimensions

<u>Example:</u> Consider the line given by:

$$\underbrace{\begin{matrix} x+y+z=0 \\ 2y+3z=0 \end{matrix}}_{L}$$, find a matrix A with kernel this line.

$$L = \left\{ t \cdot \begin{bmatrix} 1/2 \\ -3/2 \\ 1 \end{bmatrix} \middle| \; t \text{ real number} \right\}$$

$\mathbb{R}^m$ $\qquad\qquad$ $\mathbb{R}^n$



$m=3$ $\qquad\qquad$ $n \geq 2$

$$\underbrace{\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\text{equality giving } \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \text{ is in } \ker(A)} \longleftrightarrow \begin{matrix} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x+2y+3z=0$ $\quad$ ✓ plane in 3 dimensions.

$\begin{bmatrix} 1 & 2 & 3 & | & 0 \end{bmatrix}$ $\qquad$ $x = -2t - 3s$ $\qquad$ $\begin{bmatrix} -2t-3s \end{bmatrix}$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{free} & \text{free} \\ t & s \end{array} \qquad \begin{bmatrix} \vdots \\ t \\ s \end{bmatrix} \quad t, s \text{ real numbers}$$

$$\begin{bmatrix} -2t-3s \\ t \\ s \end{bmatrix} = t \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A augmented
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \end{array}\right]$$

$$\begin{array}{c} \uparrow \\ \text{free} \end{array}$$

Example: The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ has image the line $\left\{ t \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \,\middle|\, t \text{ real} \right\}$

$$a \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent if none of them can be obtained as a linear combination of the others.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_i = a_1 \vec{v}_1 + \cdots + a_{i-1} \vec{v}_{i-1} + a_{i+1} \vec{v}_{i+1} + \cdots + a_n \vec{v}_n$$

$$0 = a_1 \vec{v}_1 + \cdots + a_{i-1} \vec{v}_{i-1} - \vec{v}_i + a_{i+1} \vec{v}_{i+1} + \cdots + a_n \vec{v}_n$$

some are non-zero

Vectors are linearly independent if when:

$$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0} \quad \text{then} \quad a_1 = \cdots = a_n = 0.$$

Suppose: $a_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$, then
$$\begin{aligned} a_1 + a_2 + a_3 &= 0 \\ a_1 + a_2 &= 0 \end{aligned}$$

$$\begin{bmatrix}\vdots\\1\end{bmatrix} \quad \begin{bmatrix}\vdots\\0\end{bmatrix} \quad \begin{bmatrix}\vdots\\0\end{bmatrix} \begin{bmatrix}\vdots\\0\end{bmatrix} \qquad a_1 \qquad = 0$$

This has <u>one</u> solution: $a_1 = 0$, $a_2 = 0$, $a_3 = 0$.

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \begin{bmatrix}4\\5\\6\end{bmatrix} \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}2\\3\\4\end{bmatrix}$$, are these vectors linearly independent?

$a_1 = 2$ $\quad a_2 = -1$ $\quad a_3 = 0$ $\quad a_4 = 1$

$$\underset{0}{a_1}\begin{bmatrix}1\\1\\1\end{bmatrix} + \underset{0}{a_2}\begin{bmatrix}4\\5\\6\end{bmatrix} + \underset{0}{a_3}\begin{bmatrix}1\\2\\3\end{bmatrix} + \underset{0}{a_4}\begin{bmatrix}2\\3\\4\end{bmatrix} = \underset{0}{\begin{bmatrix}0\\0\\0\end{bmatrix}} \leftrightsquigarrow \begin{bmatrix}1&4&1&2\\1&5&2&3\\1&6&3&4\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Never the case. $\longrightarrow$ (i) No solution.

linear independence $\longrightarrow$ (ii) One solution. (*)

linear dependence $\longrightarrow$ (iii) Infinite solutions.

$$\overset{\text{rref}}{\leftrightsquigarrow} \left[\begin{array}{cccc|c}1&4&1&2&0\\1&5&2&3&0\\1&6&3&4&0\end{array}\right]$$

(*) Will be $a_1 = \cdots = a_n = 0$.

$$2\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} + \begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad 2\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&4&1&2&0\\1&5&2&3&0\\1&6&3&4&0\end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{cccc|c}1&4&1&2&0\\0&1&1&1&0\\0&2&2&2&0\end{array}\right] \xrightarrow[R_3-2R_2]{R_1-4R_2} \begin{array}{c}\begin{array}{cccc}a_1&a_2&a_3&a_4\end{array}\\ \left[\begin{array}{cccc|c}1&0&-3&-2&0\\0&1&1&1&0\\0&0&0&0&0\end{array}\right]\\ \begin{array}{cccc}&&\uparrow&\uparrow\\&&\text{free}&\text{free}\end{array}\end{array}$$

$$\begin{bmatrix}3a_3+2a_4\\-a_3-a_4\\a_3\\a_4\end{bmatrix} \qquad \begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$$

$a_4 = 1$
$a_3 = 0$

<u>Basis:</u> V a subspace of $\mathbb{R}^n$

A <u>basis</u> of V is a collection of vectors that span V and are linearly independent.

Example: Give a basis for the image of $A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 1 & 5 & 2 & 3 \\ 1 & 6 & 3 & 4 \end{bmatrix}$.

Answer: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Answer: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. Answer: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

$$\mathrm{im}(A) = \left\{ a \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \;\middle|\; a,b,c,d \text{ real} \right\} =$$

definition of span.

definition of span.

$$= \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 2 & 0 \\ 1 & 5 & 2 & 3 & 0 \\ 1 & 6 & 3 & 4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{rank } 2 \; (*)$$

free free ← remove them!

rank 2 (*) two vectors in the basis

$$\mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)$$

Given vectors $\vec{v}_1, \dots, \vec{v}_n$, their span is the set of all their linear combinations:

$$\mathrm{span}(\vec{v}_1, \dots, \vec{v}_n) = \{ a_1 \vec{v}_1 + \dots + a_n \vec{v}_n \mid a_1, \dots, a_n \text{ any real numbers} \}$$

Image of $B = \begin{bmatrix} 2 & 4 & 1 & 1 \\ 3 & 5 & 2 & 1 \\ 4 & 6 & 3 & 1 \end{bmatrix}$.

$$\mathrm{im}(B) = \mathrm{span}\underbrace{\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)}_{\text{not basis}} = \mathrm{span}\underbrace{\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right)}_{\text{not basis}}$$

L line given by
$$\mathrm{span}\Big(\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{basis of } L} \Big) = \mathrm{span}\Big(\underbrace{\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}}_{\text{basis of } L} \Big) = \mathrm{span}\Big(\underbrace{\begin{bmatrix} \pi \\ \pi \\ \pi \end{bmatrix}}_{\text{basis of } L} \Big) = \mathrm{span}\Big(\underbrace{\begin{bmatrix} \log 2 \\ \log 2 \\ \log 2 \end{bmatrix}}_{\text{basis of } L} \Big)$$

$$4 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

zero vector is <u>never</u> in a basis

$a \cdot \begin{bmatrix} \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

so $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in $\text{span}\left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right)$

$L = \text{span}\left(\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}}_{\text{not a basis}} \right)$ is living in $\mathbb{R}^3$.

The span of the vectors $\vec{v}_1, \dots, \vec{v}_n$ is a subspace.

$\text{span}(\vec{v}_1, \dots, \vec{v}_n) \leftarrow$ subspace.

A basis is a collection of vectors $\vec{w}_1, \dots, \vec{w}_m$.

We can take the span of a basis: $\text{span}(\vec{w}_1, \dots, \vec{w}_m)$.

Vector space: $\mathbb{R}^n$.

The columns of A span $\text{im}(A)$.

The kernel of A tells you if the columns are a basis.

$$a_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 4 & 1 & 2 \\ 1 & 5 & 2 & 3 \\ 1 & 6 & 3 & 4 \end{bmatrix}}_{A} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\ker(A)$

If $\ker(A)$ has <u>one</u> vector, then the columns of A are linearly independent.

Finding whether some vectors are linearly independent is equivalent to finding the kernel of the matrix having those vectors as columns.

Given a basis $\vec{v}_1, \dots, \vec{v}_n$, vectors in $\text{span}(\vec{v}_1, \dots, \vec{v}_n)$ can be written in a unique way as a linear combination:

$$\vec{v} = c_1 \cdot \vec{v}_1 + \cdots + c_n \vec{v}_n$$

unique.

We call $c_1, \dots, c_n$ the <u>coordinates</u> of $\vec{v}$ in terms of $\vec{v}_1, \dots, \vec{v}_n$.

V kernel $\longleftarrow$ subspaces $\longrightarrow$ image W

$\vec{v}_1, \dots, \vec{v}_n \longleftarrow$ basis $\longrightarrow \vec{w}_1, \dots, \vec{w}_m$

$V = \text{span}(\vec{v}_1, \dots, \vec{v}_n)$ $\qquad$ $W = \text{span}(\vec{w}_1, \dots, \vec{w}_m)$

<u>Problem 2.1.14.:</u>

(b) $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$, for which k does the matrix $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ have integer entries?

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix} \cdot \frac{1}{2k-15}$$

<u>Long way:</u> $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2k-15} \begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix}$, this gives four equations in terms of the integers a, b, c, d. Solve for k.

Note:

$$\underbrace{-b-d}_{\text{both integers}} = \frac{3}{2k-15} - \frac{2}{2k-15} = \frac{1}{2k-15}$$

So $\frac{1}{2k-15} = n$ some integer.

Now: $\frac{1}{n} = 2k-15$ so $k = \frac{15}{2} + \frac{1}{\ldots} = 7.5 + \ldots$

Now: $\frac{1}{n} = 2k - 15$ so $k = \frac{15}{2} + \frac{1}{2n} = 7.5 + \frac{1}{2n}$.

$(*)$ $nk = \dfrac{k}{2k-15}$

and: $\underbrace{\dfrac{k}{2k-15}}_{\text{is an integer.}} = 7.5n + \frac{1}{2}$

$(*)$ $nk = 7.5n + \frac{1}{2}$

So n should be odd. Then: $k = \frac{15}{2} + \frac{1}{2n}$ for n odd integer.

We found that if the entries are integers then k has this form.

We now check that if k has this form then the entries are integers:

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1} = \frac{1}{2k-15}\begin{bmatrix} k & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{2\left(\frac{15}{2}+\frac{1}{2n}\right)-15}\begin{bmatrix} \frac{15}{2}+\frac{1}{2n} & -3 \\ -5 & 2 \end{bmatrix} =$$

$k = \frac{15}{2} + \frac{1}{2n}$

$$= \frac{1}{15 + \frac{1}{n} - 15}\begin{bmatrix} \frac{15}{2}+\frac{1}{2n} & -3 \\ -5 & 2 \end{bmatrix} = n \cdot \begin{bmatrix} \frac{15}{2}+\frac{1}{2n} & -3 \\ -5 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{15n}{2}+\frac{1}{2} & -3n \\ -5n & 2n \end{bmatrix} = \begin{bmatrix} 7n + \frac{n}{2} + \frac{1}{2} & -3 \\ -5 & 2n \end{bmatrix}$$

Practice Midterm 1, Problem 6:

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}$ $\quad \vec{v}$ in $\mathbb{R}^3$

$\vec{x} \longmapsto \vec{x} \cdot \vec{v} = T(\vec{x})$

1. If $\vec{v} = \vec{0}$ then $T(\vec{x}) = \vec{x} \cdot \vec{0} = \vec{0}$ for all $\vec{x}$ in $\mathbb{R}^3$.

$\ker(T) = \mathbb{R}^3$ basis $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

$\operatorname{im}(T) = \{\vec{0}\}$ basis $\{\}$.

2. If $\vec{v} \neq \vec{0}$ then $\operatorname{im}(T) = \mathbb{R}$ with basis $\{[1]\}$.

$T(\vec{v}) = \vec{v}\cdot\vec{v} \neq 0$ in $\mathbb{R}$.

$\ker(T) = \{\vec{x} \text{ in } \mathbb{R}^3 \mid \vec{x}\cdot\vec{v} = 0\}$, let $\vec{v} = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ then:

$0 = \vec{x}\cdot\vec{v} = v_1x_1 + v_2x_2 + v_3x_3 \leftarrow$ plane with normal vector $\vec{v}$.

2.1. If $v_1 = v_2 = 0$ then $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ are in $\ker(T)$, they are linearly independent, so they form a basis.

2.2. If $v_1 = 0$ then $\begin{bmatrix}x_1 & 0\\x_2 & 1\\x_3 & -\frac{v_2}{v_3}\end{bmatrix}$ and $\begin{bmatrix}x_1 & 1\\x_2 & 1\\x_3 & -\frac{v_2}{v_3}\end{bmatrix}$ are in $\ker(T)$,

$0 = 0\cdot 0 + v_2\cdot 1 + v_3\cdot x_3$ $\qquad$ $0 = 0\cdot 1 + v_2\cdot 1 + v_3\cdot x_3$

$-\frac{v_2}{v_3} = x_3$ $\qquad$ $-\frac{v_2}{v_3} = x_3$

and they are linearly independent, so they form a basis.

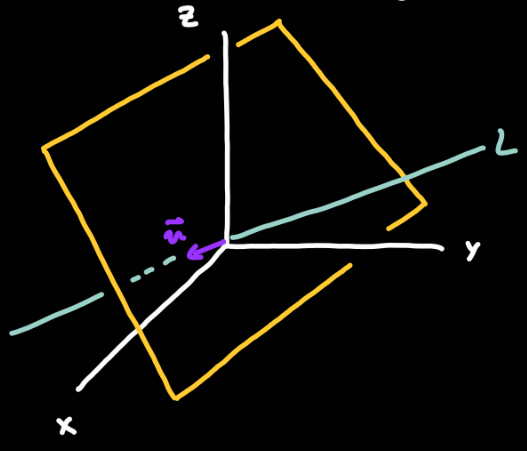
2.3. If v_1, v_2, v_3 are all non-zero then $\begin{bmatrix}-\frac{v_2}{v_3}\\1\\0\end{bmatrix}, \begin{bmatrix}-\frac{v_3}{v_1}\\0\\1\end{bmatrix}$ work.

$0 = v_1\cdot x_1 + v_2\cdot 1 + v_3\cdot 0$ $\qquad$ $0 = v_1\cdot x_1 + v_2\cdot 0 + v_3\cdot 1$

$x_1 = -\frac{v_2}{v_1}$ $\qquad$ $x_1 = -\frac{v_3}{v_1}$

Practice Midterm 1, Problem 4.:

The matrix associated to T is given by: $\left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \right]$.



$\underbrace{y = z}_{V}$ $\quad y - z = 0$ $\quad \begin{bmatrix}0\\1\\-1\end{bmatrix}$ gives L

vector perpendicular to V

$$\text{ref}_V(\vec{x}) = \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x})$$

$$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\cdot\vec{u}$$

$$\vec{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$\text{proj}_V(\vec{x}) = \vec{x} - \text{proj}_L(\vec{x})$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} - \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix} - \left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix} =$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} - \left(\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix} - \left(\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix} =$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} - \left(\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix} - \left(\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\-1\end{bmatrix} =$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}.$$