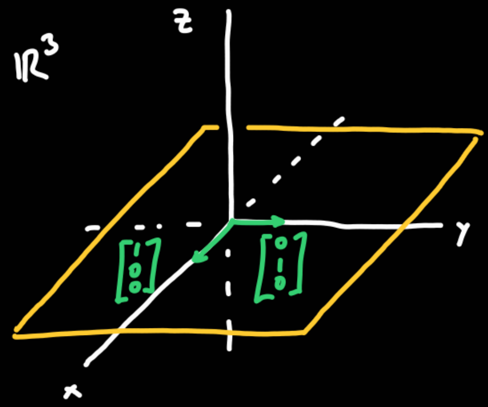
Recall: basis $\mathbb{R}^3$



$\checkmark$ $z=0$

$$V = \text{span}\Big(\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{linearly independent}}\Big)$$

$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ form a basis of V.

$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\3\\0\end{bmatrix}$ form a basis of V.

$\mathbb{R}^{n}$ has a standard basis: $\Bigg\{\underbrace{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}}_{n \text{ vectors}}\Bigg\} = S.$

Given a vector subspace, it always has a basis, and the number of elements is fixed.

The dimension of a vector subspace is the number of elements (vectors) in a basis.

$\mathbb{R}^n$ has dimension n.

Rank-Nullity: Given A an $n \times m$ matrix then:

$$\underbrace{\dim(\ker(A))}_{\text{nullity of } A} + \underbrace{\dim(\text{im}(A))}_{\dim(\text{im}(A)) = \text{rank}(A)} = m.$$

Given a basis of a vector subspace, we can write vectors (in the subspace) as a linear combination of the basis elements in a unique way:

$\mathbb{R}^2$

$$\begin{bmatrix}-1\\4\end{bmatrix} = \underline{1} \cdot \begin{bmatrix}1\\1\end{bmatrix} + \underline{1} \cdot \begin{bmatrix}-2\\3\end{bmatrix}$$

$\mathbb{R}$ unique coefficients

form a basis of $\mathbb{R}^2$

These (unique) coefficients are called the <u>coordinates</u> of the vector in the basis.

$$\boxed{\text{The vector } \begin{bmatrix}-1\\4\end{bmatrix} \text{ has coefficients } 1,1 \text{ in the basis } \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}.}$$

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix} \right\}$$

the same meaning!

$$\boxed{\begin{bmatrix}-1\\4\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}1\\1\end{bmatrix}}$$

$$\begin{bmatrix}-1\\4\end{bmatrix}_S = \begin{bmatrix}-1\\4\end{bmatrix} = 1\cdot\begin{bmatrix}1\\1\end{bmatrix} + 1\cdot\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}1 & -2\\1 & 3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \underbrace{\begin{bmatrix}1 & -2\\1 & 3\end{bmatrix}}\begin{bmatrix}-1\\4\end{bmatrix}_{\mathcal{B}}$$

matrix given by having the vectors of $\mathcal{B}$ for columns

$$\begin{bmatrix}-1\\4\end{bmatrix} = -1\cdot\begin{bmatrix}1\\0\end{bmatrix} + 4\cdot\begin{bmatrix}0\\1\end{bmatrix} \quad \text{so} \quad \begin{bmatrix}-1\\4\end{bmatrix}_S = \begin{bmatrix}-1\\4\end{bmatrix}$$

S

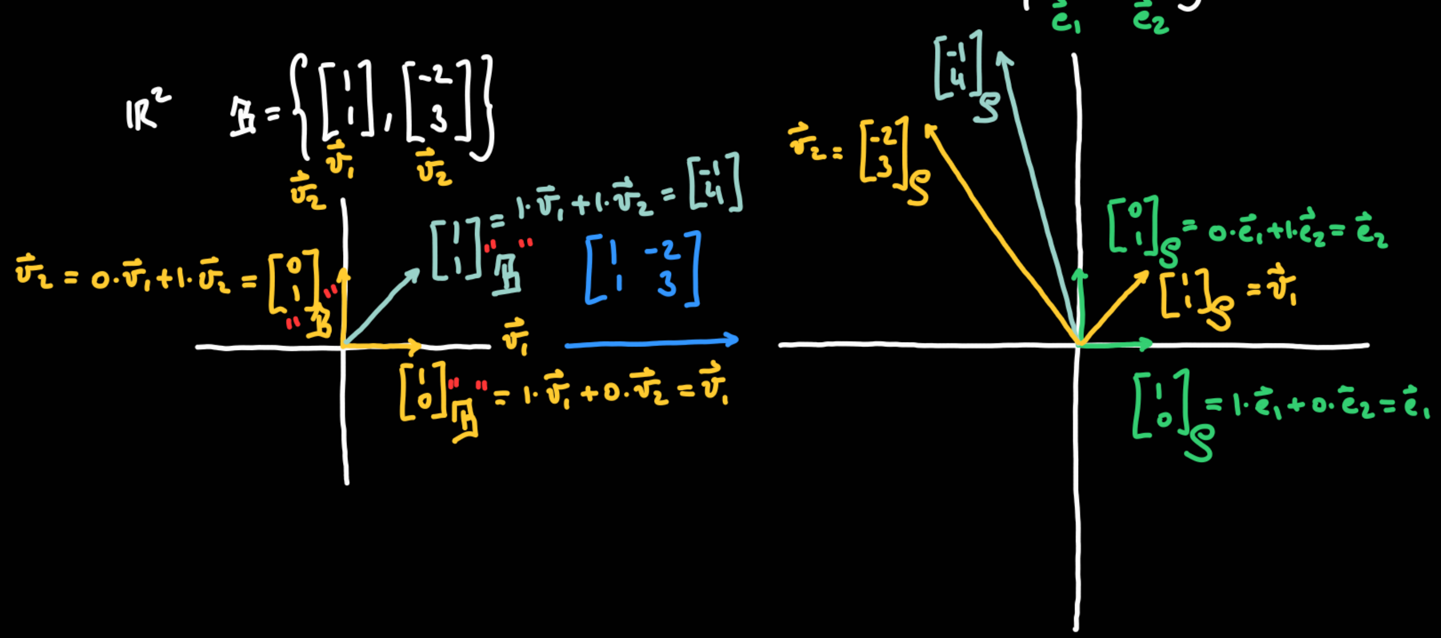
The matrix $\begin{bmatrix}1 & -2\\1 & 3\end{bmatrix}$ inputs vectors in the basis $\mathcal{B}$ and outputs vectors in the basis S.

<u>Questions:</u>

(a) Given $\mathcal{B}$ find coordinates. $\begin{bmatrix}-1\\2\end{bmatrix}, \begin{bmatrix}1\\-8\end{bmatrix}$, $\begin{bmatrix}1\\1\end{bmatrix}_S$, find $\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}}$

(b) Given coordinates, find $\mathcal{B}$. $\begin{bmatrix}2\\3\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}4\\5\end{bmatrix}$, $\begin{bmatrix}2\\-1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}1\\-1\end{bmatrix}$

$\mathbb{R}^2$ $S = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$



Given a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$, the matrix $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ inputs coefficients in $\mathcal{B}$ and outputs coefficients in $\mathcal{S}$.

If $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ is invertible, then $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}^{-1}$ inputs coefficients in $\mathcal{S}$ and output coefficients in $\mathcal{B}$.

<u>Recall:</u> Given a linear transformation $T : \overset{\mathcal{S}}{\mathbb{R}^3} \longrightarrow \overset{\mathcal{S}}{\mathbb{R}^2}$ then the matrix associated to T is:

$$\left[\underbrace{T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}_{\text{vector in } \mathbb{R}^2} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right] = A.$$

$$\begin{bmatrix} * \\ * \end{bmatrix} \quad \begin{bmatrix} * \\ * \end{bmatrix} \quad \begin{bmatrix} * \\ * \end{bmatrix} \qquad \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}_{2 \times 3}$$

Given a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, its associated matrix is:

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\overset{\mathcal{B}}{B} = \left[[T(\vec{v}_1)]_{\mathcal{B}} \quad [T(\vec{v}_2)]_{\mathcal{B}} \quad [T(\vec{v}_3)]_{\mathcal{B}} \right].$$

Now a vector with coordinates in $\mathcal{B}$ is sent to its image with coordinates in $\mathcal{B}$.

$\mathcal{S}$ $\qquad \mathbb{R}^3 \xrightarrow{\;T\;} \mathbb{R}^3$

$\qquad\qquad \vec{x} \longmapsto T(\vec{x}) = A\vec{x}$

$\mathcal{B}$ $\qquad [\vec{x}]_{\mathcal{B}} \longmapsto [T(\vec{x})]_{\mathcal{B}} = B\,[\vec{x}]_{\mathcal{B}}.$

Example: In $\mathcal{S}$ consider the orthogonal projection onto $\operatorname{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = L.$

$\mathcal{S}$ $\qquad T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad A = \left[T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)\; T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right] = \begin{bmatrix}1&0\\0&0\end{bmatrix}$

$\qquad\qquad \begin{bmatrix}x\\y\end{bmatrix} \longmapsto \begin{bmatrix}x\\0\end{bmatrix}$

$\mathcal{B} = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}\right\}$, consider $\vec{v} = \begin{bmatrix}a\\b\end{bmatrix}_{\mathcal{B}} = a\cdot\begin{bmatrix}1\\1\end{bmatrix} + b\cdot\begin{bmatrix}-2\\3\end{bmatrix}$, we want $T(\vec{v})$ in the basis $\mathcal{B}$.

$\begin{bmatrix}-1\\4\end{bmatrix} = \underbrace{\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{B}}}_{\text{coefficients in } \mathcal{B}} \iff 1\cdot\begin{bmatrix}1\\1\end{bmatrix} + 1\cdot\begin{bmatrix}-2\\3\end{bmatrix}$

$$\left[\left[T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)\right]_{\mathcal{B}} \;\left[T\left(\begin{bmatrix}-2\\3\end{bmatrix}\right)\right]_{\mathcal{B}}\right] = \left[\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}} \;\begin{bmatrix}-2\\0\end{bmatrix}_{\mathcal{B}}\right] = \begin{bmatrix}3/5 & -6/5\\ -1/5 & 2/5\end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} = a\cdot\begin{bmatrix}1\\1\end{bmatrix} + b\cdot\begin{bmatrix}-2\\3\end{bmatrix}$$

$$\begin{bmatrix}-2\\0\end{bmatrix} = c\cdot\begin{bmatrix}1\\1\end{bmatrix} + d\cdot\begin{bmatrix}-2\\3\end{bmatrix}$$

$$\left.\begin{aligned}\begin{bmatrix}1\\0\end{bmatrix} &= S\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}}\\ \begin{bmatrix}-2\\0\end{bmatrix} &= S\begin{bmatrix}-2\\0\end{bmatrix}_{\mathcal{B}}\end{aligned}\right\} \rightarrow \left.\begin{aligned}S^{-1}\begin{bmatrix}1\\0\end{bmatrix} &= \begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}}\\ S^{-1}\begin{bmatrix}-2\\0\end{bmatrix} &= \begin{bmatrix}-2\\0\end{bmatrix}_{\mathcal{B}}\end{aligned}\right\} \longrightarrow \begin{aligned}\begin{bmatrix}1\\0\end{bmatrix}_{\mathcal{B}} &= \begin{bmatrix}3/5\\-1/5\end{bmatrix}\\ \begin{bmatrix}-2\\0\end{bmatrix}_{\mathcal{B}} &= \begin{bmatrix}-6/5\\2/5\end{bmatrix}\end{aligned}$$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$$S = \begin{bmatrix}1&-2\\1&3\end{bmatrix} \qquad S^{-1} = \frac{1}{5}\begin{bmatrix}3&2\\-1&1\end{bmatrix}$$

Now $\left[T\left(\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\begin{bmatrix}-1\\4\end{bmatrix}_{\mathcal{B}}}\right)\right]_{\mathcal{B}} = \begin{bmatrix}3/5 & -6/5\\ -1/5 & 2/5\end{bmatrix}\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\begin{bmatrix}-1\\4\end{bmatrix}_{\mathcal{B}}} = \begin{bmatrix}-3/5\\1/5\end{bmatrix}$

$T\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix} \qquad \left[T\left(\begin{bmatrix}-1\\4\end{bmatrix}\right)\right]_{\mathcal{B}} = S^{-1}\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}3/5 & -6/5\\ \cdots\end{bmatrix}\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-3/5\\1/5\end{bmatrix}.$

The change of basis matrix is given by taking the vectors of $\mathfrak{B}$ and putting them in columns (forming a square matrix).

$$
\begin{tikzcd}
\mathcal{S} & \mathbb{R}^2 \arrow[rrr, "\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}"] & & & \mathbb{R}^2 \arrow[ddd, "S^{-1}"] \\
& & \begin{bmatrix} -1 \\ 4 \end{bmatrix} \arrow[r, mapsto] & \begin{bmatrix} -1 \\ 0 \end{bmatrix} \arrow[d, mapsto] & \\
& & \begin{bmatrix} 1 \\ 1 \end{bmatrix} \arrow[u, mapsto] \arrow[r, mapsto] & \begin{bmatrix} -3/5 \\ 1/5 \end{bmatrix} & \\
\mathfrak{B} & \mathbb{R}^2 \arrow[uuu, "S"] \arrow[rrr, "\begin{bmatrix} 3/5 & -6/5 \\ -1/5 & 2/5 \end{bmatrix}"'] & & & \mathbb{R}^2
\end{tikzcd}
$$