<u>Recall:</u> $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ $\qquad S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$\boxed{\begin{bmatrix} -1 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \qquad \boxed{\begin{bmatrix} -1 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix}}$$

$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix}_{\mathcal{B}}$$

$$\boxed{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}$$

$$\vec{x} = c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n$$

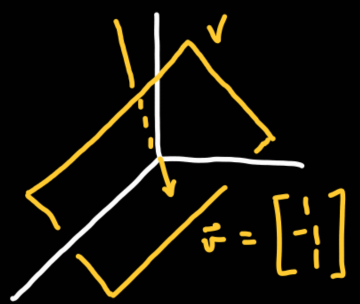
$$\mathcal{B} = \{ \vec{v}_1, \ldots, \vec{v}_n \}$$

$$S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \text{then} \quad \boxed{\vec{x} = S\,[\vec{x}]_{\mathcal{B}}}$$

<u>Example:</u> $\mathbb{R}^3$ $\qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is given by orthogonal projection on the plane

spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.



$$\mathcal{R} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\vec{x} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ in the basis } \mathcal{R}$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{R}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{R}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We want to work with basis that make our life easy.

<u>Orthonormal</u> basis: all vectors are perpendicular to each other, and all vectors have length one.

<u>Length:</u> $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

<u>Perpendicular:</u> $\vec{v}$ is perpendicular to $\vec{w}$ when $\vec{v}\cdot\vec{w} = 0$.

<u>orthogonal</u> basis: all vectors are perpendicular to each other.

Orthonormal vectors are always linearly independent.

$\mathbb{R}^n$ if we find $\vec{v}_1, \dots, \vec{v}_n$ that are linearly independent, they form a basis.

$\mathbb{R}^n$ if we find $\vec{v}_1, \dots, \vec{v}_n$ that are orthonormal, they form a basis.

Given an orthonormal basis, computing projections is easy:

$\underbrace{\{\vec{u}_1, \dots, \vec{u}_n\} = \mathcal{B}}_{\text{basis of some subspace } V}$, given $\vec{x}$ we can project it onto V as:

$$\mathrm{proj}_V(\vec{x}) = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{x}\cdot\vec{u}_n)\vec{u}_n$$

$$\left[\mathrm{proj}_V(\vec{x})\right]_{\mathcal{B}} = \begin{bmatrix} \vec{x}\cdot\vec{u}_1 \\ \vdots \\ \vec{x}\cdot\vec{u}_n \end{bmatrix}$$

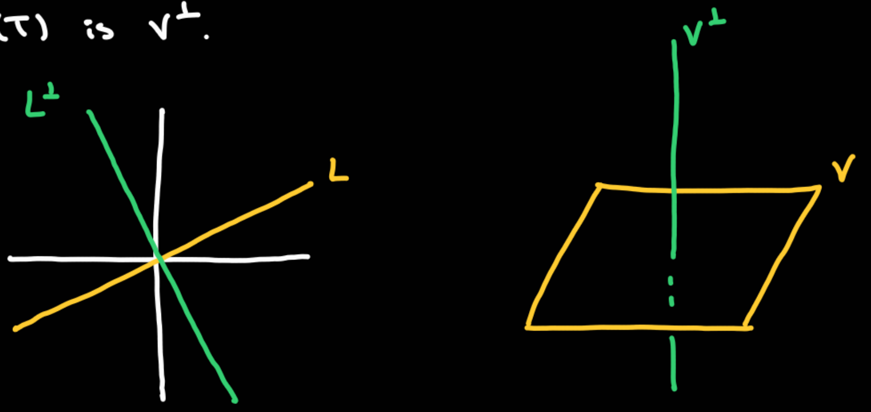
<u>Orthogonal complement:</u> V vector subspace of $\mathbb{R}^n$

$V^\perp$ is the <u>orthogonal complement</u> of V

$$V^\perp = \{\vec{x} \text{ in } \mathbb{R}^n \mid \vec{x}\cdot\vec{v} = 0 \text{ for all } \vec{v} \text{ in } V\}$$

Consider $T: \mathbb{R}^n \to \mathbb{R}^n$ given by orthogonal projection onto V, then

$\ker(T)$ is $V^\perp$.



Example: Consider $\mathbb{R}^3$, $V = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\1\end{bmatrix}\right)$, find an orthonormal basis of V.

Note that:

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\1\end{bmatrix} = -2+1+1 = 0$$ so these vectors are perpendicular.

Thus:

$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}$ are an orthonormal basis of V.

Find an orthonormal basis of $\mathbb{R}^3$ with $\vec{u}_1, \vec{u}_2$ as elements of the basis.

We can do this by finding a vector $\vec{u}$ perpendicular to both $\vec{u}_1, \vec{u}_2$.

$$\vec{u} = \begin{bmatrix}x\\y\\z\end{bmatrix} \qquad \begin{aligned}\vec{u}\cdot\vec{u}_1 &= 0\\ \vec{u}\cdot\vec{u}_2 &= 0\end{aligned}$$

Note that $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is perpendicular to V. Thus $\vec{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$ is perpendicular

to $\vec{u}_1$ and $\vec{u}_2$, and has length one.

$\mathcal{B} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$.

The matrix:

$$\begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$ has image V. $\qquad \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

Given a vector subspace V in $\mathbb{R}^n$, the intersection of V and $V^\perp$ is just $\vec{0}$.

Also: $\dim(V) + \dim(V^\perp) = n$.

Also: $(V^\perp)^\perp = V$.

Given $\vec{x}, \vec{y}$ in $\mathbb{R}^n$, then:

$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ when $\vec{x}$ and $\vec{y}$ are perpendicular.

$\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$ with equality if and only if $\vec{x}$ is in V.

$$\cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \cdot \|\vec{y}\|}.$$