Recall: $\vec{x}\cdot\vec{y}=0$ if and only if $\vec{x}$ and $\vec{y}$ are perpendicular

orthogonal basis

orthonormal basis

Gram–Schmidt process: given a basis $\mathcal{B}$, produces an orthonormal basis.

Example: $\mathbb{R}^3$, $\mathcal{B}=\left\{\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\vec{v}_1},\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\vec{v}_2},\underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{\vec{v}_3}\right\}$.

$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$ has rank 3

Consider $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ given by projection onto $V=\operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right)$.

$T(\vec{x})=\operatorname{proj}_V(\vec{x})=\vec{x}-(\vec{x}\cdot\vec{u})\vec{u}$ with $\vec{u}$ perpendicular to V
unitary

$\|\vec{u}\|=1$

$\left.\begin{array}{l}\vec{u}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}=0\\ \vec{u}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}=0\end{array}\right\}$ give the line perpendicular to V.

$$\vec{u}=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix}-\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}=$$

$$=\begin{bmatrix}1\\0\\0\end{bmatrix}-\frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}2/3\\1/3\\-1/3\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix}-\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}=$$

$$=\begin{bmatrix}0\\1\\0\end{bmatrix}+\frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} - \left(\begin{bmatrix}0\\0\\1\end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}-1/3\\1/3\\2/3\end{bmatrix}$$

$$A = \left[\, T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \;\; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \;\; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) \right] = \begin{bmatrix}2/3 & 1/3 & -1/3\\ 1/3 & 2/3 & 1/3\\ -1/3 & 1/3 & 2/3\end{bmatrix}$$

$$B = \left[\, [T(\vec{v}_1)]_{\mathfrak{B}} \;\; [T(\vec{v}_2)]_{\mathfrak{B}} \;\; [T(\vec{v}_3)]_{\mathfrak{B}} \right] = \begin{bmatrix}1 & 1/3 & 0\\ 0 & 0 & 0\\ 0 & 1/3 & 1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix} - \left(\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} = 1 \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + 0 \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\\1\end{bmatrix} = 0 \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + 1 \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1/3\\2/3\\1/3\end{bmatrix} = \frac{1}{3} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + 0 \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} + \frac{1}{3} \cdot \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$B\,[\vec{x}]_{\mathfrak{B}} = [T(\vec{x})]_{\mathfrak{B}} \qquad V = \operatorname{span}\left(\underset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\1\\1\end{bmatrix}}\right)$$

$$A\,[\vec{x}]_S = [T(\vec{x})]_S$$

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix} \rightsquigarrow [\vec{x}]_S = S\,[\vec{x}]_{\mathfrak{B}}$$

$$\mathfrak{B} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

Is this an orthogonal basis? <u>NO</u>.

How do we get an orthonormal basis from $\mathcal{B}$?

1. Make the first vector unitary. $(\vec{v}_1)$

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

2. Extract the perpendicular component of $\vec{v}_2$ with respect to $\vec{u}_1$.

$$L = \text{span}(\vec{u}_1) \qquad \text{proj}_L(\vec{v}_2) = (\vec{v}_2\cdot\vec{u}_1)\,\vec{u}_1$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2\cdot\vec{u}_1)\,\vec{u}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \left(\begin{bmatrix}1\\0\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} =$$

$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} \qquad \|\vec{v}_2^{\perp}\| = \sqrt{1+\frac{2}{4}} = \sqrt{\frac{3}{2}}$$

4 should be a 4

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\\sqrt{2}\end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

3. Find the component of $\vec{v}_3$ perpendicular to both $\vec{u}_1$ and $\vec{u}_2$.

$$V = \text{span}(\vec{u}_1, \vec{u}_2)$$

$$\vec{v}_3^{\parallel} = \text{proj}_V(\vec{v}_3) = (\vec{v}_3\cdot\vec{u}_1)\,\vec{u}_1 + (\vec{v}_3\cdot\vec{u}_2)\,\vec{u}_2$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\parallel} = \vec{v}_3 - (\vec{v}_3\cdot\vec{u}_1)\,\vec{u}_1 - (\vec{v}_3\cdot\vec{u}_2)\,\vec{u}_2 =$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} - \left(\begin{bmatrix}0\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} - \left(\begin{bmatrix}0\\1\\1\end{bmatrix}\cdot\frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}\right)\frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} =$$

$$= \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\cdot\frac{1}{2}\cdot\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}-\frac{1}{6}\\1-\frac{1}{2}+\frac{1}{6}\\1-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}-2/3\\2/3\\2/3\end{bmatrix}$$

$$= \frac{\vec{v}_3^{\perp}}{\ldots} = \frac{\sqrt{3}}{\ldots}\begin{bmatrix}-2/3\\\ldots\end{bmatrix} = \ldots\begin{bmatrix}-1\\\ldots\end{bmatrix} \qquad \|\vec{v}_3^{\perp}\| = 2\ldots$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{3}{2}\begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \|\vec{v}_3^{\perp}\| = \frac{2}{\sqrt{3}}$$

$\mathcal{R} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal basis

This process is encoded in the <u>QR-decomposition</u>:

$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} = QR = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

upper triangular

Q has orthonormal columns

$$Q = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}$$

$$r_{11} = \|\vec{v}_1\| \qquad r_{ii} = \|\vec{v}_i^{\perp}\|$$

$$r_{ij} = \vec{u}_i \cdot \vec{v}_j$$

To do the QR decomposition of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ we need to compute:

$$r_{11} = \|\vec{v}_1\| = \sqrt{2} \qquad r_{22} = \|\vec{v}_2^{\perp}\| = \sqrt{\frac{3}{2}} \qquad r_{33} = \|\vec{v}_3^{\perp}\| = \frac{2}{\sqrt{3}}$$

$$r_{12} = \vec{u}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$r_{13} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$r_{23} = \vec{u}_2 \cdot \vec{v}_3 = \frac{\sqrt{2}}{\sqrt{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \sqrt{\frac{2}{3}} \cdot \left(-\frac{1}{2} + 1\right) = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{2} = \frac{1}{\sqrt{6}}$$

$$R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{\mathcal{B}} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & \sqrt{2/3} & 1/\sqrt{3} \end{bmatrix}}_{Q}\underbrace{\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}}_{R}$$

should be

$R \quad \sim \frac{2}{\sqrt{6}}$

Orthogonal transformations and matrices:

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orthogonal if it preserves lengths: $\|T(\vec{x})\| = \|\vec{x}\|$.

Orthogonal transformations preserve angles (and in particular they preserve orthogonality).

$\mathcal{B}$ orthonormal $\rightsquigarrow \{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ will be an orthonormal basis.

A linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is orthogonal if and only if $\{T(\vec{e}_1), \dots, T(\vec{e}_n)\}$ is an orthonormal basis.