Recall: $\mathcal{B} = \left\{ \underset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\0\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\1\\1\end{bmatrix}} \right\}$ $\qquad V = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$ $\qquad \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$

$$\mathcal{R} = \left\{ \underset{\vec{u}_1}{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}}, \underset{\vec{u}_2}{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}}, \underset{\vec{u}_3}{\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}} \right\}$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} R$$

QR decomposition

R is not a change of basis matrix!

Orthogonal matrices/transformations: preserve lengths, angles, ...

Given $\underset{\vec{v}}{\mathcal{B}}$ and $\underset{\vec{u}}{\mathcal{R}}$ basis, then there is an invertible matrix S:

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = S^{-1} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} S \quad , \quad S \text{ is the change of basis matrix.}$$

$$= \underbrace{(S^{-1})}_{S} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} \underbrace{(S^{-1})^{-1}}_{S^{-1}} \qquad S = S^{-1}$$

Example: of orthogonal matrices. (there are no orthonormal matrices)

1. Given M invertible, then Q in the QR-decomposition of M is orthogonal.

$$M = QR$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}}_{Q \text{ is orthogonal}} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$$

2. 2×2 orthogonal matrix:

$$\underbrace{\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\frac{1}{\sqrt{13}}}_{A} \qquad \underbrace{\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\frac{1}{\sqrt{13}}}_{A^{-1}} \qquad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I_2} \qquad I_2^{-1} = I_2$$

The columns form a basis.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The columns are orthogonal.

The columns are <u>not</u> unitary. We can fix this by dividing by $\sqrt{13}$.

3. $B = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$ is an orthogonal matrix.

$$B^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} = B^T$$

$$B = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix} \qquad B^T = \begin{bmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \\ - & \vec{u}_3 & - \end{bmatrix} \qquad B^T B = \begin{bmatrix} \vec{u}_1\cdot\vec{u}_1 & \vec{u}_1\cdot\vec{u}_2 & \vec{u}_1\cdot\vec{u}_3 \\ \vec{u}_2\cdot\vec{u}_1 & \vec{u}_2\cdot\vec{u}_2 & \vec{u}_2\cdot\vec{u}_3 \\ \vec{u}_3\cdot\vec{u}_1 & \vec{u}_3\cdot\vec{u}_2 & \vec{u}_3\cdot\vec{u}_3 \end{bmatrix}$$

orthonormal basis

$$= I_3$$

<u>Taking transposes:</u> $A : \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad A^T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$A : \mathbb{R}^m \to \mathbb{R}^n \qquad A^{-1} : \mathbb{R}^n \to \mathbb{R}^m$

This "looks like" transposing!

(i) $(A+B)^T = A^T + B^T$

(iii) $(kA)^T = k(A^T)$

(iii) $(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1} A^{-1}$

(iv) $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

(v) $(A^T)^{-1} = (A^{-1})^T$

Note that if Q is an invertible orthogonal matrix then $Q^{-1} = Q^T$.

Let V be a subspace of $\mathbb{R}^n$ with orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$. Then the matrix encoding the projection of $\mathbb{R}^n$ onto V is given by: $\dim(V) = m$

$$P = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} - & \vec{u}_1 & - \\ & \vdots & \\ - & \vec{u}_m & - \end{bmatrix}}_{Q^T}$$

Q is invertible if and only if $m = n$.

P is an $n \times n$ matrix.

Example: $V = \operatorname{span}\left(\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec{w}}\right)$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

Compute the matrix associated to projecting onto V.

$$\vec{u}_1 = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{w}^\perp = \vec{w} - (\vec{w} \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

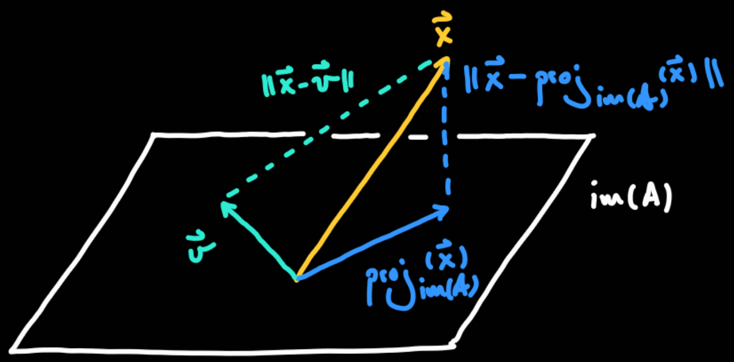
$$\vec{u}_2 = \frac{\vec{w}^\perp}{\|\vec{w}^\perp\|} = \underbrace{\frac{2}{\sqrt{6}}}_{\frac{1}{\|\vec{w}^\perp\|}} \cdot \underbrace{\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}_{\vec{w}^\perp} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Now:

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

this is exactly the projection on the basis S.

Least squares approximation:



$\|\vec{x}-\mathrm{proj}_{im(A)}(\vec{x})\| < \|\vec{x}-\vec{v}\|$

for $\vec{v}$ in $im(A)$ <u>not</u>

$\mathrm{proj}_{im(A)}(\vec{x})$.

A vector $\vec{x}^*$ is a <u>least-squares solution</u> of the system $A\vec{x}=\vec{b}$ if and only if $\|\vec{b}-A\vec{x}^*\| \le \|\vec{b}-A\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^n$.

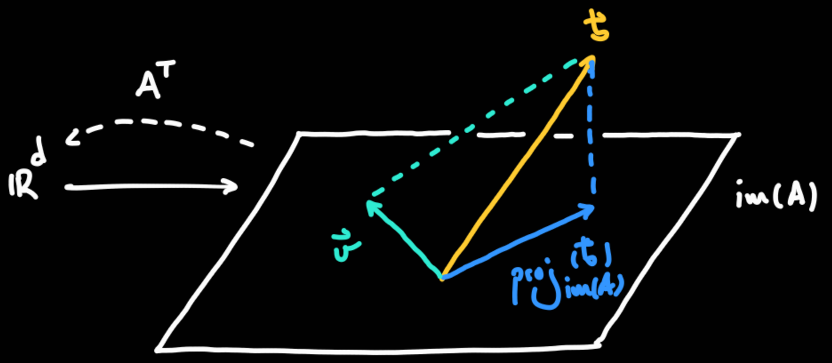
<u>Remark:</u>

(i) $\left(im(A)\right)^{\perp} = ker(A)^T$.

(ii) $ker(A) = ker(A^TA)$

In particular if $ker(A)=\{\vec{0}\}$ then $ker(A^TA)=\{\vec{0}\}$, and when A is a projection then A^TA will be square.

Then A^TA will be invertible (equivalently $rref(A^TA) = I_n$).



$A\vec{x} = \vec{b}$ $A\vec{x}^* = \text{proj}(\vec{b}) \rightsquigarrow \vec{x}^* = A^{-1}\,\text{proj}(\vec{b})$

no solution

$A\vec{x} = \vec{b} \rightsquigarrow A^TA\vec{x} = A^T\vec{b}$ normal equation of $A\vec{x} = \vec{b}$

always has a solution!

If $\ker(A) = \{\vec{0}\}$ then $A\vec{x} = \vec{b}$ has a unique least-squares solution:

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}.$$

$$(A^TA)^{-1} \neq A^{-1}(A^T)^{-1}$$

⚠

Example: $V = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$ with $\vec{v} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}0\\1\\1\end{bmatrix}$. Find the least squares solution of:

$$\underbrace{\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}}_{A}\vec{x} = \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\vec{b}}$$

$\text{im}(A) = V$ $\vec{b}$ is not in $\text{im}(A)$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b} = \cdots = \frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix}.$$

$$A\vec{x}^* = \frac{1}{3}\begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{3}\vec{v} + \frac{1}{3}\vec{w}.$$

$$P\vec{b} = \text{proj}_{\text{im}(A)}(\vec{b}) = \frac{1}{3}\vec{v} + \frac{1}{3}\vec{w}.$$