Determinants:

$\det : \underbrace{M_n(\mathbb{R})}_{} \longrightarrow \mathbb{R}$ satisfying:

all square matrices of size $n \times n$ and real entries

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(i) linear on the columns. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1+1 \\ 1 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \cdot 1 \\ 1 & 2 \cdot 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

NOT

$\det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \overset{⚠}{=} \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \det \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

(ii) alternating in the columns: if two columns are equal, then the determinant is zero. $\det \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 2 \\ 3 & 6 & 3 \end{bmatrix} = 0$

(iii) the determinant of I_n is 1. $\det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = 1$

$n=2$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has determinant $ad-bc$.

$$\det \begin{bmatrix} a+a' & b \\ c+c' & d \end{bmatrix} = \underbrace{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{} + \underbrace{\det \begin{bmatrix} a' & b \\ c' & d \end{bmatrix}}_{}$$

$$(a+a')d - b(c+c') = ad + a'd - bc + bc' = \underbrace{ad-bc}_{} + \underbrace{a'd-bc'}_{}$$

$n=3$: Sarrus Rule

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{33}a_{21}a_{12} - a_{32}a_{23}a_{11} - a_{31}a_{22}a_{13}.$$

<u>Triangular matrices:</u> the determinant is the product of the diagonal elements.

Given a matrix A, we compute $\mathrm{ref}(A)$. To do this we swapped rows s times, and we divided by the scalars $k_1, \dots, k_r$. Then:

$$\det(A) = (-1)^s \cdot k_1 \cdots k_r.$$

<u>Example:</u> Compute $\det\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

$s = 0$, $k_1 = -4$, $k_2 = 2$ determinant is -8.

$s = 1$, $k_1 = 4$, $k_2 = 2$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$1\cdot2\cdot2 + 2\cdot1\cdot2 + 3\cdot3\cdot1$$
$$- 2\cdot3\cdot2 - 1\cdot1\cdot1 - 2\cdot2\cdot3 =$$
$$= 4 + 4 + 9 - 12 - 1 - 12 =$$
$$= 17 - 25 = -8$$

$A \xrightarrow{\text{swap}} B$ $\det(A) = -\det(B)$

$A \xrightarrow{\text{add rows}} B$ $\det(A) = \det(B)$

$A \xrightarrow{\text{multiply } k} B$ $\det(A) = \frac{1}{k}\det(B)$

$$\det(A) \xrightarrow[\text{swap}]{=} -\det(B) \xrightarrow[\text{divide } 4]{=} -4\cdot\det(C) \xrightarrow[\text{divide } 2]{=} -4\cdot2\cdot\det(D) \quad -8$$

$$\mathrm{ref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: Compute $\det\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix}$.

Expand along a column:

$$\det\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix} = (-1)^{1+1}\cdot 1\cdot \underbrace{\det\underbrace{\begin{bmatrix}2&1\\1&2\end{bmatrix}}_{\text{submatrices}}}_{\text{minor (real number)}} + (-1)^{2+1}\cdot 3\cdot\det\begin{bmatrix}2&3\\1&2\end{bmatrix} + (-1)^{3+1}\cdot 2\cdot\det\begin{bmatrix}2&3\\2&1\end{bmatrix} =$$

1st column

$$= 1\cdot(4-1) - 3\cdot(4-3) + 2\cdot(2-6) = 3-3-8 = -8.$$

Expand along a row:

$$\det\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix} = (-1)^{1+1}\cdot 1\cdot\det\begin{bmatrix}2&1\\1&2\end{bmatrix} + (-1)^{1+2}\cdot 2\cdot\det\begin{bmatrix}3&1\\2&2\end{bmatrix} + (-1)^{1+3}\cdot 3\cdot\det\begin{bmatrix}3&2\\2&1\end{bmatrix} =$$

1st row

$$= 1\cdot(4-1) - 2\cdot(6-2) + 3\cdot(3-4) = 3-8-3 = -8.$$

Given A, what is $\det(A^T)$? $\det(A^T) = \det(A)$

$$\begin{bmatrix}1&2&3\\3&2&1\\2&1&2\end{bmatrix} = A \qquad A^T = \begin{bmatrix}1&3&2\\2&2&1\\3&1&2\end{bmatrix} \longleftarrow \text{expand along 1st row}$$

expand along 1st column

$$\begin{bmatrix}2&3\\1&2\end{bmatrix} \qquad \begin{bmatrix}2&1\\3&2\end{bmatrix}$$