<u>Recall:</u> $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ $\qquad A = [a_{ij}]$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(\underbrace{A_{ij}}) \qquad \text{j-th column (expansion)}$$

matrix obtained by removing i-th row and j-th column from A.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(A_{ij}) \qquad \text{i-th row (expansion)}$$

Note that the determinant is:

(i) linear in rows. $\qquad$ (i) linear in columns.

(ii) alternating in rows. $\qquad$ (ii) alternating in columns.

(iii) $\det(I_n) = 1$. $\qquad$ (iii) $\det(I_n) = 1$.

As a consequence $\det(A^T) = \det(A)$.

(1) If B is obtained from A by taking a row of A and dividing it by k, then: $\qquad$ determinant is linear

$$A = \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} \qquad B = \begin{bmatrix} - & \frac{\vec{r}_1}{k} & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix}$$

$$\det(B) = \frac{1}{k} \cdot \det \begin{bmatrix} - & \vec{r}_1 & - \\ & \vdots & \\ - & \vec{r}_n & - \end{bmatrix} = \frac{1}{k} \cdot \det(A)$$

(2) If B is obtained from A by swapping two rows then:

$\det(B) = -\det(A)$. $\qquad$ determinant is linear and alternating

(3) If B is obtained from A by taking a row of A and adding it to a row of A, then:

$$\det(B) = \det(A).$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad -8 \qquad \begin{array}{c} \begin{array}{ccc} \log(2) & \log(2) & \log(2) \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{array} \\ \begin{bmatrix} 1+2\pi & 2+\pi & 3+2\pi \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \end{array} \rightsquigarrow -8$$

Use linearity and alternating to explain this.

$$\begin{bmatrix} 1+2\pi & 2+\pi & 3+2\pi \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1+2\pi & 2+\pi & 3+2\pi \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2\pi & \pi & 2\pi \end{bmatrix}$$

$$\det \begin{bmatrix} 1+2\pi & 2+\pi & 3+2\pi \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} + \underbrace{\det \begin{bmatrix} 2\pi & \pi & 2\pi \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}}_{\pi \cdot \underbrace{\det \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}}_{0}}$$

A square matrix is invertible if and only if the determinant is <u>not</u> zero.

Suppose A is invertible (so $\det(A) \neq 0$), can $\det(A^{-1}) = 0$?

<u>No!</u> A^{-1} is invertible, so $\det(A^{-1}) \neq 0$.

What is $\det(A^{-1})$? Which real number is $\det(A^{-1})$?

$$\det(A^{-1}) = \frac{1}{\det(A)} = \det(A)^{-1}$$

determinant is multiplicative

Given A, B square matrices, then $\det(A \cdot B) = \det(A) \cdot \det(B)$.

<u>Example:</u> There are matrices A, B such that $\det(A+B) \neq \det(A) + \det(B)$.

$$\det\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right) = \det\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} = 8$$

$$\det\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} = 0 \qquad \det\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = 4$$

Let n be even, $A = I_n$, $B = -I_n$.

$\det(A) = 1$, $\det(B) = (-1)^n = 1$ so $\det(A) + \det(B) = 2$.

n even

$\det(A+B) = \det(0) = 0$.

<u>Example:</u> Compute A^{-1} and $\det(A^{-1})$ for $A = \begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 1 & 2\end{bmatrix}$.

$$\begin{bmatrix}1 & 2 & 3\\ 3 & 2 & 1\\ 2 & 1 & 2\end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \xrightarrow{\frac{1}{-4}R_2} \xrightarrow[R_3+3R_2]{R_1-2R_2} \xrightarrow{\frac{1}{2}R_3} \xrightarrow[R_2-2R_3]{R_1+R_3} \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ -2 & 0 & 1\end{bmatrix} \begin{bmatrix}1 & 0 & 0\\ 3/4 & -1/4 & 0\\ -2 & 0 & 1\end{bmatrix} \begin{bmatrix}-1/2 & 1/2 & 0\\ 3/4 & -1/4 & 0\\ 1/4 & -3/4 & 1\end{bmatrix} \begin{bmatrix}-1/2 & 1/2 & 0\\ 3/4 & -1/4 & 0\\ 1/8 & -3/8 & 1/2\end{bmatrix}$$

$$\begin{bmatrix}-3/8 & 1/8 & 1/2\\ 1/2 & 1/2 & -1\\ 1/8 & -3/8 & 1/2\end{bmatrix} = A^{-1} \qquad \det(A^{-1}) = \frac{1}{2} \cdot \frac{1}{-4} = \frac{1}{-8} = \frac{1}{\det(A)}.$$

Properties:

(i) $\det(kA) = \det\begin{bmatrix} | & & | \\ k\vec{c}_1 & \cdots & k\cdot\vec{c}_n \\ | & & | \end{bmatrix} = k^n \cdot \det(A)$

(ii) $\det(A^m) = \det(A)^m$

(iii) If A is similar to B then $\det(A) = \det(B)$.

$B = S^{-1}AS \rightsquigarrow \det(B) = \det(S^{-1}AS) = \det(S^{-1}) \cdot \det(A) \cdot \det(S) =$

$= \frac{1}{\det(S)} \cdot \det(A) \cdot \det(S) = \det(A)$

Are there matrices A, B such that $A = 4\, S^{-1} B S$ for some invertible matrix S? $A = 4 \cdot I_n \quad B = I_n$?

$\det(A) = \det(4 \cdot S^{-1} \cdot B \cdot S) = 4^n \cdot \det(S^{-1} B S) = 4^n \cdot \det(B)$

Practice Midterm 2, Problem 1, (e):

$A, S \qquad S^{-1}AS = 2A.$

Least squares: Given $A, \vec{b}$, find the least squares solution $\vec{x}^*$: $\vec{s}$

It satisfies: $A\vec{x}^* = \text{proj}_{\text{im}(A)}(\vec{b})$. $\qquad A\vec{s} = \text{proj}_{\text{im}(A)}(\vec{b})$

This system will always have a solution. $\qquad \vec{s} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

If $\ker(A) = \{\vec{0}\}$ then $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$.

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ find the least squares solution of $A\vec{x} = \vec{b}$.

Find $\vec{s}$ such that $A\vec{s} = \text{proj}_{\text{im}(A)}(\vec{b})$.

The projection matrix onto $\text{im}(A)$ is: $P = A(A^TA)^{-1}A^T$

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\text{proj}_{\text{im}(A)}(\vec{b}) = P\vec{b} = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

We are solving $A\vec{s} = \text{proj}_{\text{im}(A)}(\vec{b})$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} \qquad \begin{cases} x = \frac{1}{3} \\ x + y = \frac{2}{3} \\ y = \frac{1}{3} \end{cases} \qquad \vec{s} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$