Given Q an orthogonal matrix, what is $\det(Q)$?

$\det(Q) = \det(Q^T)$ $\quad |\det(Q)| = 1$ $\quad \det(Q) = \pm 1.$

$Q^{-1} = Q^T$ $\quad QQ^{-1} = I_n$ $\quad QQ^T = I_n$ $\quad \det(QQ^T) = \det(I_n) = 1$

$$\det(QQ^T) = \det(Q)\cdot\det(Q^T) = \det(Q)^2$$

So $\det(Q)^2 = 1$. So $\det(Q) = \pm 1$.

<u>Geometric interpretation:</u>

Given $A = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$ a 2×2 matrix, its QR-decomposition is: $A = QR$.

Q orthogonal matrix

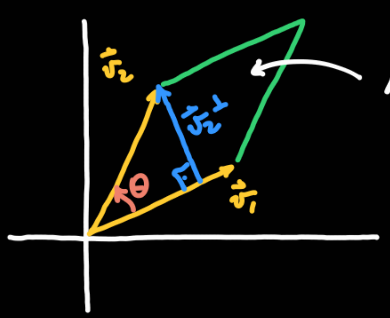
R upper triangular

$r_{11} = \|\vec{v}_1\|$ $\quad r_{12} = \vec{u}_1 \cdot \vec{v}_2$

$r_{21} = 0$ $\quad r_{22} = \|\vec{v}_2^{\perp}\|$

$\det(A) = \det(QR) = \det(Q)\cdot\det(R)$

$$|\det(A)| = \underbrace{|\det(Q)|}_{1}\cdot|\det(R)| = |\det(R)| = \underbrace{\|\vec{v}_1\|\cdot\|\vec{v}_2^{\perp}\|}_{\text{area of the parallelogram spanned by } \vec{v}_1 \text{ and } \vec{v}_2.}$$



Area of the parallelogram spanned by $\vec{v}_1$ and $\vec{v}_2$

$$\text{base}\cdot\text{height} = \underbrace{\|\vec{v}_1\|}_{\text{base}}\cdot\underbrace{\|\vec{v}_2^{\perp}\|}_{\text{height}} = \|\vec{v}_1\|\cdot\|\vec{v}_2\|\cdot\sin(\theta)$$

$\|\vec{v}_2^{\perp}\| = \|\vec{v}_2\|\cdot\sin(\theta)$

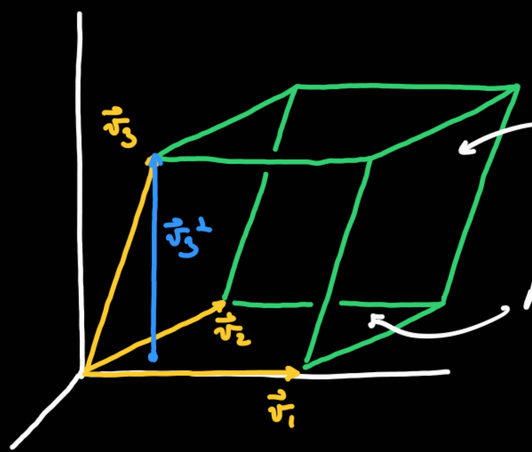
In more generality, if $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ then $A = QR$ has:

Q orthogonal matrix

R upper triangular

$r_{11} = \|\vec{v}_1\|, \quad r_{22} = \|\vec{v}_2^{\perp}\|, \ldots, \quad r_{nn} = \|\vec{v}_n^{\perp}\|$

$$|\det(A)| = |\det(QR)| = \underbrace{|\det(Q)|}_{1} \cdot |\det(R)| = \underbrace{\|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| \cdots \|\vec{v}_n^{\perp}\|}_{\text{Area of the parallelogram spanned by } \vec{v}_1, \ldots, \vec{v}_n}.$$



Volume = base · height $= \|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\| \cdot \|\vec{v}_3^{\perp}\|$.

Area base: base · height $= \|\vec{v}_1\| \cdot \|\vec{v}_2^{\perp}\|$

<u>Cramer's Rule:</u> $A\vec{x} = \vec{b}$, A invertible, then there is a closed formula for the solutions:

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}.$$

$A_{\vec{b},i}$ is the matrix obtained by swapping the i-th column of A with $\vec{b}$.

<u>Theorem:</u> Given A invertible then $A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$.

The <u>classical adjoint</u> of A is the matrix with entries:

$$(\operatorname{adj}(A))_{ij} = (-1)^{i+j} \cdot \det(A_{ji})$$

submatrix of A obtained by removing the j-th row and the i-th column.

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Step 1: Compute all minors in their respective spots/places:

$$\begin{bmatrix} (-1)^{1+1} \cdot \det(A_{11}) & (-1)^{1+2} \cdot \det(A_{12}) & (-1)^{1+3} \cdot \det(A_{13}) \\ (-1)^{2+1} \cdot \det(A_{21}) & (-1)^{2+2} \cdot \det(A_{22}) & (-1)^{2+3} \cdot \det(A_{23}) \\ (-1)^{3+1} \cdot \det(A_{31}) & (-1)^{3+2} \cdot \det(A_{32}) & (-1)^{3+3} \cdot \det(A_{33}) \end{bmatrix} = \begin{bmatrix} 3 & -4 & -1 \\ -1 & -4 & 3 \\ -4 & 8 & -4 \end{bmatrix}$$

$$(-1)^{i+j} \cdot \det(A_{ij})$$

Step 2: Take the transpose:

$$\begin{bmatrix} 3 & -1 & -4 \\ -4 & -4 & 8 \\ -1 & 3 & -4 \end{bmatrix} \qquad (-1)^{i+j} \cdot \det(A_{ji}) = (\operatorname{adj}(A))_{ij}$$

Step 3: Compute $\det(A)$ and divide:

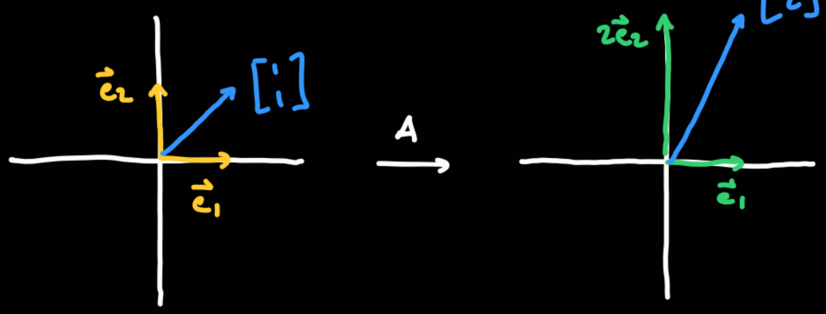
Recall that $\det(A) = -8$, so: $$A^{-1} = \begin{bmatrix} -3/8 & 1/8 & 1/2 \\ 1/2 & 1/2 & -1 \\ 1/8 & -3/8 & 1/2 \end{bmatrix}$$

Note that not all directions are the same for a linear transformation.

Example:

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

(1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$



A direction is "preferred" if the linear transformation doesn't change it.

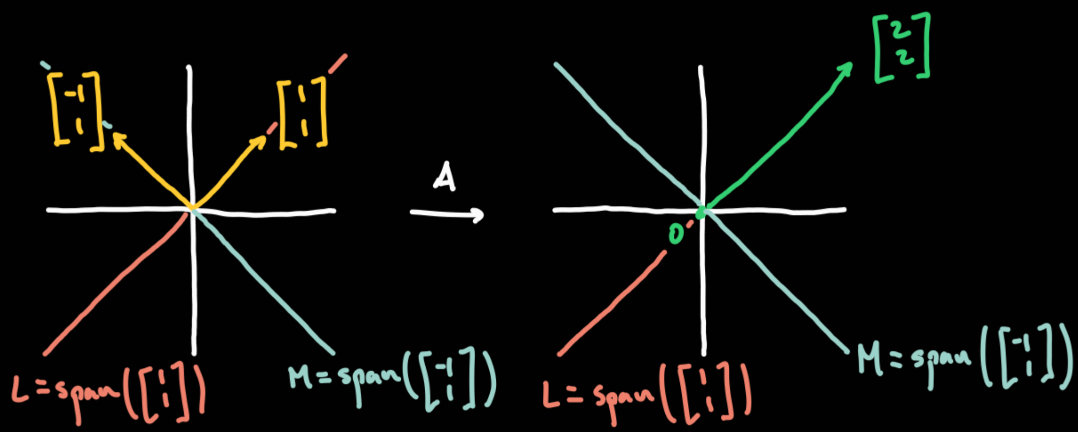
Here $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has preferred directions $\vec{e}_1$ and $\vec{e}_2$ with scaling factors 1 and 2 respectively.

(2) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Is there a vector $\vec{v}$ in $\mathbb{R}^2$ such that the direction of $\vec{v}$ is preserved by A? (one line has two orientations)

$$A\left(t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = t \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = (2t) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.



Do this! A is an orthogonal projection onto the line L followed by a scaling of 2.

If $\vec{v}$ is such that $\vec{v} \in \ker(A)$ then:

belongs in

$$A(k\vec{v}) = k\cdot(A\vec{v}) = k\cdot\vec{0} = \vec{0}$$ so $k\vec{v} \in \ker(A)$ so $\operatorname{span}(\vec{v})$ is a subspace of $\ker(A)$.

(3) $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ Does A have preferred directions?

A is symmetric $A = A^T$.

A is the orthogonal projection onto $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, with perpendicular vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

So A has three preferred directions:

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ with scaling factors $1, 1, 0$.

$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(4) Rotations have <u>no</u> preferred directions.

$(\theta \neq 0, \pi)$