<u>Recall:</u> We claimed that there was a basis $\mathcal{B}$ such that the linear transformations had a diagonal matrix associated to them.

(1) $A=\begin{bmatrix}1&0\\0&2\end{bmatrix}$ (2) $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ (3) $A=\begin{bmatrix}2/3&1/3&-1/3\\1/3&2/3&1/3\\-1/3&1/3&2/3\end{bmatrix}$.

Find one basis for each linear transformation (satisfying the above).

(1) $\mathcal{S}=\{\vec{e}_1,\vec{e}_2\}$ $\quad B=\left[\,\left[A\begin{bmatrix}1\\0\end{bmatrix}\right]_{\mathcal{S}}\;\left[A\begin{bmatrix}0\\1\end{bmatrix}\right]_{\mathcal{S}}\,\right]=\begin{bmatrix}1&0\\0&2\end{bmatrix}$.

eigenvectors of $\begin{bmatrix}1&0\\0&2\end{bmatrix}$

(2) $\mathcal{B}=\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ $\quad B=\left[\,\left[A\begin{bmatrix}1\\1\end{bmatrix}\right]_{\mathcal{B}}\;\left[A\begin{bmatrix}-1\\1\end{bmatrix}\right]_{\mathcal{B}}\,\right]=\begin{bmatrix}2&0\\0&0\end{bmatrix}$

eigenvectors of $\begin{bmatrix}1&1\\1&1\end{bmatrix}$

(3) $\mathcal{B}=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$ $\quad B=\left[\,\left[A\begin{bmatrix}1\\1\\0\end{bmatrix}\right]_{\mathcal{B}}\;\left[A\begin{bmatrix}0\\1\\1\end{bmatrix}\right]_{\mathcal{B}}\;\left[A\begin{bmatrix}1\\-1\\1\end{bmatrix}\right]_{\mathcal{B}}\,\right]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

eigenvectors of $\frac{1}{3}\begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix}$.

Given A a matrix, a non-zero vector $\vec{v}$ is called an <u>eigenvector</u> if there is a scalar λ in A such that $A\vec{v}=\lambda\vec{v}$. We say that λ is the <u>eigenvalue</u> associated to the eigenvector $\vec{v}$.

(1) $A=\begin{bmatrix}1&0\\0&2\end{bmatrix}$ has eigenvectors $\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}$ with associated eigenvalues 1, 2 respectively.

(2) $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ has eigenvectors $\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\1\end{bmatrix}$ with eigenvalues 2, 0 respectively.

(3) $A=\frac{1}{3}\begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix}$ has eigenvectors $\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\end{bmatrix}$ with eigenvalues 1, 1, 0 respectively.

Question: What are the possible eigenvalues of an orthogonal matrix? Q

If $\vec{v}$ is an eigenvector of eigenvalue λ, then:

$$Q\vec{v} = \lambda\vec{v} \qquad \|Q\vec{v}\| = \|\vec{v}\|$$

$$\underbrace{\|\vec{v}\|}_{\vec{v} \neq \vec{0}} = \|Q\vec{v}\| = \|\lambda\vec{v}\| = |\lambda| \cdot \underbrace{\|\vec{v}\|}_{\vec{v} \neq \vec{0}} \quad \text{so} \quad 1 = |\lambda| \quad \text{so} \quad \lambda = \pm 1.$$

① Given A an $n \times n$ matrix with eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ (eigenvalues $\lambda_1, \dots, \lambda_n$). If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ form a basis of $\mathbb{R}^n$ then:

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \cdot A \cdot \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}.$$

If we understand (enough) eigenvectors, we understand the matrix.

② Given A an $n \times n$ matrix such that:

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}^{-1} \cdot A \cdot \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

If we understand the matrix then we understand the eigenvectors.

where $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$. Then $\vec{v}_1, \dots, \vec{v}_n$ are eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_n$ respectively.

eigenvalue

$$A\vec{v} = \lambda\vec{v} \qquad A\vec{v} - \lambda\vec{v} = 0 \qquad (A - \lambda \cdot I_n)\cdot\vec{v} = 0 \qquad (\vec{v} \text{ in } \ker(A - \lambda \cdot I_n)).$$

eigenvector

If $\det(A - \lambda \cdot I_n) \neq 0$ there are <u>no</u> solutions. So we need $\det(A - \lambda \cdot I_n) = 0$.

The <u>characteristic equation</u> of A is: $\det(A - \lambda \cdot I_n) = 0$.

The <u>characteristic polynomial</u> of A is: $f_A(x) = \det(A - x \cdot I_n)$.

A real number λ is an eigenvalue of A if and only if $f_A(\lambda) = 0$.

<u>Example</u>: Find (by factoring the characteristic polynomial) the eigenvalues of:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

$$f_A(x) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - x \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \det \begin{bmatrix} 1-x & 1 \\ 1 & 1-x \end{bmatrix} = (1-x)^2 - 1 =$$

$$= 1 + x^2 - 2x - 1 = x^2 - 2x = x \cdot (x-2)$$

<u>Example</u>: Find $f_A(x)$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$f_A(x) = \det \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix} = (a-x)(d-x) - bc = x^2 - dx - ax + ad - bc =$$

$$= x^2 - \underbrace{(a+d)}_{\text{sum of diagonal elements}} x + \underbrace{(ad-bc)}_{\det(A)}$$

The sum of the diagonal elements of a matrix A is called the <u>trace</u> of A, denoted $\mathrm{tr}(A)$.

Given a matrix A and an eigenvalue λ, the <u>algebraic multiplicity</u> of λ is the largest integer k such that: $f_A(x) = (x-\lambda)^k \cdot g(x)$ with $g(\lambda) \neq 0$.

<u>Example</u>: Find the algebraic multiplicities of the eigenvalues of $\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

$$f_A(x) = \det \begin{bmatrix} 2/3 - x & 1/3 & -1/3 \\ 1/3 & \frac{2}{3} - x & 1/3 \\ -1/3 & 1/3 & \frac{2}{3} - x \end{bmatrix} = \cdots = -x^3 + 2x^2 - x = -\underset{0}{x}^{1}(\underset{1}{x-1})^{2}.$$

A has eigenvalues $0, 1$ with multiplicities $1, 2$.

<u>Question:</u> Let A be $n \times n$. How many distinct eigenvalues can A have?

$f_A(x)$ is a polynomial of degree n, so it has at most n solutions.
(eigenvalues)

So A has at most n distinct eigenvalues.

<u>Example:</u> $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad f_A(x) = \det \begin{bmatrix} x & -1 \\ 1 & x \end{bmatrix} = \underbrace{x^2 + 1}_{\text{no solutions}}.$

<u>Example:</u> Let A be a matrix of odd size. Does it have eigenvalue(s)?

$n \times n$ $\quad$ n odd