<u>Recall:</u> Eigenvalues are the roots of the characteristic polynomial.

$$A\vec{v} = \lambda\vec{v} \qquad A\vec{v} - \lambda\vec{v} = 0 \qquad (A - \lambda\cdot I_n)\vec{v} = 0$$

Note that if $\vec{v}$ is an eigenvector of eigenvalue λ then $\vec{v}$ is in $\ker(A-\lambda\cdot I_n)$.

Viceversa, if $\vec{v}$ is in $\ker(A-\lambda\cdot I_n)$ then $\vec{v}$ is an eigenvector of eigenvalue λ.

The subspace $\underbrace{\ker(A-\lambda\cdot I_n)}_{E_\lambda}$ is called the <u>eigenspaces</u> of A of eigenvalue λ.

<u>Example:</u> Compute/find the eigenspaces of $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$.

The eigenvalues of A are $\underbrace{1}_{E_1}, \underbrace{0}_{E_0}$.

$E_0 = \ker(A - 0\cdot I_3) = \ker(A)$, so we have to solve $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 2/3 & 1/3 & -1/3 & 0 \\ 1/3 & 2/3 & 1/3 & 0 \\ -1/3 & 1/3 & 2/3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

so $E_0 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$. ← all eigenvectors with eigenvalue 0.

$E_1 = \ker(A - 1\cdot I_3)$, so we have to solve $\underbrace{\begin{bmatrix} -1/3 & 1/3 & -1/3 \\ 1/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & -1/3 \end{bmatrix}}_{A - I_3}\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -1/3 & 1/3 & -1/3 & 0 \\ 1/3 & -1/3 & 1/3 & 0 \\ -1/3 & 1/3 & -1/3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{so} \quad \vec{x} = \begin{bmatrix} t \\ t+s \\ s \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\cdot\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$. ← all eigenvectors of eigenvalue 1.

The <u>geometric multiplicity</u> of an eigenvalue λ is $\dim(E_\lambda) = \text{gemu}(\lambda)$.

$$\text{gemu}(\lambda) = \dim(E_\lambda) = \dim(\ker(A - \lambda \cdot I_n)) = n - \dim(\operatorname{im}(A - \lambda \cdot I_n)) =$$

$$= n - \operatorname{rank}(A - \lambda \cdot I_n).$$

Algebraic multiplicity and geometric multiplicity alone do <u>not</u> give information about the linear independence of eigenvectors.

$$\underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\text{matrix of eigenvalues}} = S^{-1} A \underbrace{S}_{\text{matrix of eigenvectors}}$$

Let A be an $n \times n$ matrix, an <u>eigenbasis</u> of A is a basis of $\mathbb{R}^n$ formed by eigenvectors of A.

<u>Example:</u>

(1) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenbasis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

(2) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenbasis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

(3) $A = \frac{1}{3}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ has eigenbasis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(4) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does <u>not</u> have an eigenbasis.

Let A be a matrix of size $n \times n$. Let λ be an eigenvalue of A. Then A has an eigenbasis if and only if $\underbrace{\text{almu}(\lambda)}_{\text{factor } \lambda} = \underbrace{\text{gemu}(\lambda)}_{n - \operatorname{rank}(A - \lambda \cdot I_n)}$ for all λ. ⚠ <u>and</u> $f_A(x)$ factors from $f_A(x)$ into linear terms.

Method to compute eigenbasis:

0. Check that there exists an eigenbasis.

1. Compute eigenvalues (solve $f_A(x) = 0$, obtain $\lambda_1, \dots, \lambda_s$).

2. Compute eigenspaces (find E_λ, $\dim(E_\lambda) = \text{gemu}(\lambda)$).

3. Find a basis of the eigenspaces.

4. Concatenate these basis:

$$\{ \underbrace{\vec{v}_1^{\lambda_1}, \dots, \vec{v}_{\text{gemu}(\lambda_1)}^{\lambda_1}}_{\text{basis of } E_{\lambda_1}}, \underbrace{\vec{v}_1^{\lambda_2}, \dots, \vec{v}_{\text{gemu}(\lambda_2)}^{\lambda_2}}_{\text{basis of } E_{\lambda_2}}, \dots, \underbrace{\vec{v}_1^{\lambda_s}, \dots, \vec{v}_{\text{gemu}(\lambda_s)}}_{\text{basis of } E_{\lambda_s}} \}$$

this is now an eigenbasis of $\mathbb{R}^n$.

Remark: We are using that eigenvectors with distinct eigenvalues are linearly independent.

$v_1 \quad v_2 \qquad A\vec{v}_1 = \lambda_1 \vec{v}_1 \qquad A\vec{v}_2 = \lambda_2 \vec{v}_2$

$v_1 = k\vec{v}_2$

Check this!

Also, $\text{gemu}(\lambda) \leq \text{almu}(\lambda) \leq n$. $\qquad$ A has eigenvalues $\lambda_1, \dots, \lambda_s$.

$f_A(x)$ has degree n.

What is the sum of the algebraic multiplicities?

$$\text{almu}(\lambda_1) + \cdots + \text{almu}(\lambda_s) \leq n.$$

If we have an eigenbasis then $\text{almu}(\lambda_1) + \cdots + \text{almu}(\lambda_s) = n$.

Also: $\text{gemu}(\lambda_1) = \text{almu}(\lambda_1), \ldots, \text{gemu}(\lambda_s) = \text{almu}(\lambda_s)$, so

$$\underbrace{\text{gemu}(\lambda_1) + \cdots + \text{gemu}(\lambda_s) = n.}_{\text{the concatenation of the eigenspaces has } n \text{ vectors}}$$

A matrix A of size $n \times n$ has an eigenbasis if and only if:

$$\text{gemu}(\lambda_1) + \cdots + \text{gemu}(\lambda_s) = n.$$

Question: Let A be an $n \times n$ matrix with n distinct eigenvalues.

Does it have an eigenbasis? What is this eigenbasis?

So $f_A(x) = (x-\lambda_1)\cdots(x-\lambda_n)$. $\quad \vec{v}_1 \ \ldots \ \vec{v}_n \longleftarrow$ associated eigenvectors $\quad \mathfrak{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$

$$A = \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 5 \end{bmatrix} \qquad f_A(x) = (x-1)(x-2)(x-3)(x-4)(x-5)\cdot(-1)$$

If A and B are similar then:

(1) $f_A(x) = f_B(x)$ so they have the same eigenvalues.

(2) $\text{rank}(A) = \text{rank}(B)$

(3) $\text{almu}_A(\lambda) = \text{almu}_B(\lambda)$

for all eigenvalues λ.

$\mathrm{gemu}_A(\lambda) = \mathrm{gemu}_B(\lambda)$

(4) $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$.

Let A be such that the sum of the geometric multiplicities is the size of
$n \times n$ $\mathrm{gemu}(\lambda_1) + \cdots + \mathrm{gemu}(\lambda_s) = n$

the matrix. We say that A is <u>diagonalizable</u>.

Let $\mathfrak{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an eigenbasis of A. Consider $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\vec{x} \longmapsto A\vec{x}$, then

the matrix associated to T in the basis $\mathfrak{B}$ is diagonal, and the diagonal

entries are eigenvalues of A.

<u>Example:</u> $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is <u>not</u> diagonalizable.

$f_A(x) = (x-1)^2$ so $\mathrm{almu}(1) = 2$.

But $E_1 = \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has dimension 1. So $\mathrm{gemu}(1) = 1$.

<u>Example:</u> $A = \begin{bmatrix} 8 & -9 \\ 4 & -4 \end{bmatrix}$, is it diagonalizable (does it have an eigenbasis)?

Eigenvalues : 2.

Algebraic multiplicity : 2.

Geometric multiplicity : 1.

<u>Example:</u> $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ find an eigenbasis and find a diagonal matrix D

similar to A.