Recall: A matrix (${n \times n}$) is diagonalizable (i.e. similar to a diagonal matrix) if and only if the sum of its geometric multiplicities is the size of the matrix (n).

Example: Find an eigenbasis and a diagonal matrix similar to:

(1) $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

$f_A(x) = (1-x)(2-x)$ so A has eigenvalues $\lambda = 1$ and $\lambda = 2$.

Since we have a 2×2 matrix with two distinct eigenvalues, A is similar to:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

$E_1 = \ker(A - I_2)$ so we have to solve $\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \vec{x} = t\cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\substack{\vec{v} \\ 1}}$

$E_2 = \ker(A - 2I_2)$ so we have to solve $\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \vec{x} = t\cdot \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\substack{\vec{w} \\ 2}}$

$\mathcal{B} = \{\vec{v}, \vec{w}\}$ $\qquad$ $\mathcal{B} = \{\vec{w}, \vec{v}\}$.

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad$ **Task:** find an eigenbasis and a diagonal matrix B similar to A.

$f_A(x) = \underline{(x-1)^2}$ so A has one eigenvalue $\lambda = 1$.

the characteristic polynomial factors into linear terms, so there is a possibility of A being diagonalizable

⚠ If the characteristic polynomial does not split into linear terms then the matrix will <u>not</u> be diagonalizable.

We compute the geometric multiplicity of $\lambda = 1$.

$$\text{gemu}(\lambda) = \dim(E_\lambda) = \text{number of elements in the (eigenbasis) of } E_\lambda.$$

$E_1 = \ker(A - I_2)$ so we solve $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \vec{x} = t \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{v}}.$

So $\text{gemu}(1) = 1$, $\text{almu}(1) = 2$, so A does not diagonalize.

(3) $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Diagonalizable: Yes. No.

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$$f_A(x) = \det(A - x \cdot I_3) = (1-x)(-x)(-x) = \underbrace{-x^2(x-1)}_{\text{factors into linear terms}}$$

$\lambda = 0$ with $\text{almu}(0) = 2$

$\lambda = 1$ with $\text{almu}(1) = 1$

$E_0 = \ker(A) \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \vec{x} = \begin{bmatrix} -t-s \\ t \\ s \end{bmatrix} = t \cdot \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + s \cdot \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}.$

$\text{gemu}(0) = 2$

$E_1 = \ker(A - I_3) \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \vec{x} = t \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{w}}$

$\text{gemu}(1) = 1$

So $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is diagonalizable!

$\mathfrak{B} = \{\overset{0}{\vec{v}_1}, \overset{0}{\vec{v}_2}, \overset{1}{\vec{w}}\}$ is an eigenbasis. $\mathfrak{R} = \{\overset{1}{\vec{w}}, \overset{0}{\vec{v}_1}, \overset{0}{\vec{v}_2}\}$ is an eigenbasis.

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

Exercise: Find a,b,c for which $\begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable. Try it

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, find A^3, A^5, A^{100}.

$$A = S^{-1} D S \qquad D \text{ diagonal}$$

$$A^2 = (S^{-1} D S)^2 = (S^{-1} D S)(S^{-1} D S) = S^{-1} D^2 S$$

$$A^{100} = S^{-1} D^{100} S.$$

$$f_A(x) = (1-x)(3-x) - 8 = x^2 - 4x - 5 = (x+1)(x-5) \qquad \lambda = -1, \quad \lambda = 5$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So A is similar to $\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$

So: $\quad B = S^{-1} A S \quad$ with $\quad S = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$, so:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = A = S B S^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

Now:

$$A^{100} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix} \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}.$$

Complex eigenvalues:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad a,b \text{ real numbers.} \quad \xrightarrow{\mathbb{C}} \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$$

Theorem (Fundamental Theorem of Algebra): Let $f(x)$ be a polynomial with coefficients in $\mathbb{C}$ of degree $n>0$. Then $f(x)$ has a root in $\mathbb{C}$.

Symmetric matrices and quadratic forms.

$A^T = A$

Can we find orthonormal eigenbasis? $\quad B = S^{-1}AS = S^TAS.$

$S^{-1} = S^T$

A is <u>diagonalizable</u> if it has an eigenbasis.

A is <u>orthogonally diagonalizable</u> if it has an orthonormal eigenbasis.

<u>Question</u>: Are there orthogonally diagonalizable matrices? <u>Yes</u>.

Have we seen any of them? <u>Yes</u>. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

<u>Theorem</u>: (Spectral Theorem) A matrix is orthogonally diagonalizable if and only if it is symmetric.

$D = \underbrace{S^{-1}AS = S^T}AS. \quad S$ orthogonal matrix

$A = SDS^T$

$(MN)^T = N^TM^T$

$A^T = (SDS^T)^T = \underbrace{(S^T)^T}_{S}\underbrace{D^T}_{D}S^T = SDS^T = A$ so A is symmetric.

We have $D = S^TAS$ if and only if $A^T = A$.

S orthogonal: $S^{-1} = S^T$

<u>not</u> $S = S^T$ necessarily

$$(SDS^T)^T = ((\underbrace{SD}_{M})\underbrace{S^T}_{N})^T = (\underbrace{S^T}_{N})^T(\underbrace{SD}_{M})^T = (S^T)^T(D^T S^T)$$

If A has distinct eigenvalues, the corresponding eigenvectors are linearly independent.

If A is symmetric and has distinct eigenvalues, the corresponding eigenvectors are perpendicular. (Recall that perpendicular implies linearly independent).