Recall: A is orthogonally diagonalizable if and only if $A = A^T$.

Method to find orthonormal eigenbasis: $A = A^T$

1. Find eigenvalues.

2. Find eigenspaces.

3. Find a basis for each eigenspace.

4. Find an orthonormal basis for each eigenspace.

5. Concatenate these basis.

Example: $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$

Question: Is A orthogonally diagonalizable?

Yes. $A = A^T$.

We want an orthonormal eigenbasis.

$\lambda = 1 \qquad \lambda = 0$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\mathcal{B} = \Bigg\{ \underbrace{\overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}}_{E_1}, \underbrace{\overset{\vec{v}_3}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}}_{E_0} \Bigg\}.$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\,\parallel} = \operatorname{proj}_{\operatorname{span}(\vec{v}_1)}(\vec{v}_2) = (\vec{v}_2^{\,\parallel} \cdot \vec{u}_1)\, \vec{u}_1 = \cdots = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{2}{\sqrt{6}}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

$$\vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\mathcal{R} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Quadratic forms:

$$q : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

q is a quadratic form when it is a linear combination of $x_i x_j$.

q does not have to be linear!

We can write any quadratic form q as $q(\vec{x}) = \vec{x}^T A \vec{x}$ for some symmetric matrix A.

Example:

$$q(x_1, x_2, x_3) = \frac{2}{3}(\overset{x_1x_1}{\underset{a_{11}}{x_1^2}} + \overset{x_2x_2}{\underset{a_{22}}{x_2^2}} + \overset{x_3x_3}{\underset{a_{33}}{x_3^2}} + \underset{\frac{a_{12}}{2}}{x_1x_2} - \underset{\frac{a_{13}}{2}}{x_1x_3} + \underset{\frac{a_{23}}{2}}{x_2x_3})$$

$$q : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$x_1x_2 = x_2x_1$

$a_{12} = a_{21}$

Question: The variables x_ix_j corresponds to which entry in matrix A?

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\underset{1\times 3}{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{1\times 1}{\frac{2}{3}(x_1^2 + x_2^2 + x_3^2 + x_1x_2 - x_1x_3 + x_2x_3)}$$

Since $\underset{n\times n}{A}$ is symmetric, there is an orthonormal eigenbasis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$.

Rewriting $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ in terms of $\mathcal{B}$, now:

$$q(\vec{x}) = \vec{x}^T A \vec{x} = (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)^T A (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) =$$

$$= (c_1\vec{v}_1^T + \cdots + c_n\vec{v}_n^T) A (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) =$$

λ_n λ_1

$$= (c_1\vec{v}_1^T + \cdots + c_n\vec{v}_n^T)(c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n) = \qquad \vec{v}_i^T\vec{v}_j = \vec{v}_i \cdot \vec{v}_j = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$= \lambda_1 c_1^2 + \cdots + \lambda_n c_n^2 .$$

<u>Example:</u> Determine whether $q(x_1, x_2, x_3) = \frac{2}{3}(x_1^2 + x_2^2 + x_3^2 + x_1x_2 - x_1x_3 + x_2x_3)$

has a minimum, maximum, or neither, at $x_1 = x_2 = x_3 = 0$.

(is global) (is global)

$$A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix} \qquad 1, 1, 0$$

$$q(c_1, c_2, c_3) = c_1^2 + c_2^2 \geq 0$$

The vector $x_1 = x_2 = x_3 = 0$ has coordinates $c_1 = c_2 = c_3 = 0$, so it is a minimum.

$c_1 = c_2 = 0$

c_3 anything.

c_3 can be as close to 0 as we want.

<u>Question:</u> Is it local? Is it global? Neither?

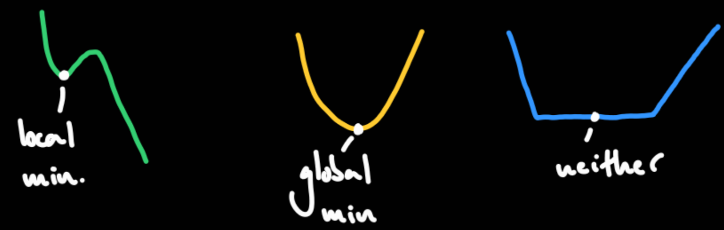
This is a weird minimum.

$\underbrace{\text{Everything around is larger.}}_{}$ 0

The minimum is never reached elsewhere. 2

We can have too many minimums next to each other. 0

The line $c_1 = c_2 = 0$ is a minimum.



Let $q(\vec{x}) = \vec{x}^T A \vec{x}$ be a quadratic form. $\quad (q(\vec{0}) = \vec{0})$

(i) A is <u>positive definite</u> if $q(\vec{x})$ is positive for all non-zero $\vec{x}$.

(ii) A is <u>positive semidefinite</u> if $q(\vec{x}) \geq 0$ for all $\vec{x}$.

(iii) A is <u>indefinite</u> if $q(\vec{x})$ takes positive and negative values.

A is positive definite if and only if all its eigenvalues are positive.

A is positive semidefinite if and only if all its eigenvalues are positive or zero.

$A^{(i)}$ is the $i \times i$ matrix obtained by deleting all rows and columns after the i-th ones.

We say that $A^{(i)}$ is the i-th principal submatrix of A.

A is positive definite if and only if all the principal submatrices have positive determinant.

<u>Example:</u> $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ is <u>not</u> positive definite.

$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

indefinite

$\det(A^{(3)}) = 0.$

$\det(A^{(1)}) = \det(2/3) = \frac{2}{3} > 0$

$\det(A^{(2)}) = \det \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \frac{1}{3} > 0$

$\det(A^{(3)}) = \det \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ \ldots \end{bmatrix} = 0$ so A is not positive definite.

$$\begin{bmatrix} \cdots \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$$

As a consequence of the Spectral Theorem, if A is symmetric then there is a basis of $\mathbb{R}^n$ made of orthogonal vectors such that when applying A the vectors remain orthogonal.

We will be able to do this in general: given A there is a basis of $\mathbb{R}^n$ made of orthonormal vectors such that when applying A they remain orthogonal.

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find an orthonormal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ such that $A\vec{v}_1$ and $A\vec{v}_2$ are orthogonal.

How do we use the Spectral Theorem?

How do we get a symmetric matrix from A?

$A^T A$ is always symmetric!

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2} \qquad \lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

$$\underbrace{\vec{v}_1 = \begin{bmatrix} \sqrt{5-\sqrt{5}}/10 \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} -\sqrt{5+\sqrt{5}}/10 \\ \sqrt{2}/\sqrt{5+\sqrt{5}} \end{bmatrix}}_{\text{form an orthonormal eigenbasis of } A^T A.}$$

Now we claim that $A\vec{v}_1$ and $A\vec{v}_2$ are orthogonal.

$$A\vec{v}_1 = \begin{bmatrix} \sqrt{1+\frac{2}{\sqrt{5}}} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix} \qquad A\vec{v}_2 = \begin{bmatrix} -\sqrt{1-\frac{2}{\sqrt{5}}} \\ 2/\sqrt{5+\sqrt{5}} \end{bmatrix}$$

$$\underset{\text{dot product}}{(A\vec{v}_1)\cdot(A\vec{v}_2)} = (A\vec{v}_1)^T(A\vec{v}_2) = \vec{v}_1^{\,T}A^TA\vec{v}_2 = \lambda_2\vec{v}_1^{\,T}v_2 = \lambda_2\underset{\text{dot product}}{\vec{v}_1\cdot\vec{v}_2} = 0.$$

$$\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y}$$