<u>Recall:</u> Given A, there is an orthonormal basis $\underset{\text{orthonormal eigenbasis of } A^TA.}{\mathcal{B}}$ such that applying A to the elements of $\mathcal{B}$ gives orthogonal vectors.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2} \qquad \lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$$

$$\begin{bmatrix} \sqrt{5-\sqrt{5}}/\sqrt{10} \\ \sqrt{2}/\sqrt{5-\sqrt{5}} \end{bmatrix} = \vec{u}_1 = \vec{v}_1 \qquad \vec{u}_2 = \vec{v}_2 = \begin{bmatrix} -\sqrt{5+\sqrt{5}}/\sqrt{10} \\ \sqrt{2}/\sqrt{5+\sqrt{5}} \end{bmatrix} \qquad \mathcal{B}$$

$A\vec{u}_1$ is perpendicular to $A\vec{u}_2$.

Moreover:

$$\|A\vec{u}_1\|^2 = \lambda_1 \quad \text{and} \quad \|A\vec{u}_2\|^2 = \lambda_2$$

<u>(*) 30-min later:</u>

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \begin{bmatrix} \sqrt{5+\sqrt{5}}/\sqrt{10} \\ \sqrt{5-\sqrt{5}}/\sqrt{10} \end{bmatrix}.$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \begin{bmatrix} -\sqrt{5-\sqrt{5}}/\sqrt{10} \\ \sqrt{5+\sqrt{5}}/\sqrt{10} \end{bmatrix}.$$

The <u>singular values</u> of a matrix $\underset{n\times m}{A}$ are the square roots of the eigenvalues of $\underset{m\times m}{A^TA}$ counted with multiplicity.

$$\sigma_1, \dots, \sigma_m \qquad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_m$$

singular values

$$\sigma_1 = \sqrt{\lambda_1}, \dots, \sigma_m = \sqrt{\lambda_m}$$

eigenvalues of A^TA.

Given A, there is an orthonormal basis $\mathcal{B}$ such that applying A to the vectors of $\mathcal{B}$ yields orthogonal vectors whose lengths are the singular values of A.

... yields orthogonal vectors whose lengths are the singular values of A.

$$A \rightsquigarrow \underset{\text{symmetric}}{A^TA} \overset{\text{WORK}}{\rightsquigarrow} \mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\} \quad \lambda_1, \dots, \lambda_m \quad \sigma_1 = \sqrt{\lambda_1}, \dots, \sigma_m = \sqrt{\lambda_m}$$

orthonormal eigenbasis

$\rightsquigarrow A\vec{v}_1, \dots, A\vec{v}_m$ are orthogonal

$$\|A\vec{v}_1\| = \sigma_1, \dots, \|A\vec{v}_m\| = \sigma_m.$$

Given $\underset{n\times m}{A}$, $r = \text{rank}(A)$. We can write $A = U \Sigma V^T$ <u>singular value dec.</u>

$\underset{n\times n}{U}$ orthogonal matrix, $\underset{n\times m}{\Sigma}$ "diagonal", $\underset{m\times m}{V}$ orthogonal.

$$U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \vdots \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & \ddots & 0 \cdots 0 \end{bmatrix} \quad V = \underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}}_{\mathcal{B} \text{ orthonormal eigenbasis of } A^TA.}$$

non-zero singular values of A.

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1, \dots, \vec{u}_r = \frac{1}{\sigma_r} A\vec{v}_r$$

we then complete to an orthonormal basis of $\mathbb{R}^n$.

$$\underbrace{\vec{u}_1, \dots, \vec{u}_r}_{\text{coming from the non-zero } \sigma_1, \dots, \sigma_r}, \underbrace{\vec{u}_{r+1}, \dots, \vec{u}_n}_{\text{basis of the complement.}}$$

$$\text{im} \underset{n\times r}{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r \\ | & & | \end{bmatrix}} \text{ subspace of } \mathbb{R}^n$$

$\mathbb{R}^r \to \mathbb{R}^n$

$$\left(\text{im}\left[\vec{u}_1 \cdots \vec{u}_r\right]\right)^\perp = \underbrace{\ker\left(\left[\vec{u}_1 \cdots \vec{u}_r\right]^T\right)}_{\text{complement of } \vec{u}_1, \dots, \vec{u}_r.}$$

(*) $$U = \begin{bmatrix} \sqrt{5+\sqrt{5}}/\sqrt{10} & -\sqrt{5-\sqrt{5}}/\sqrt{10} \\ \sqrt{5-\sqrt{5}}/\sqrt{10} & \sqrt{5-\sqrt{5}}/\sqrt{10} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{\frac{3}{2} + \frac{\sqrt{5}}{2}} & 0 \\ & \sqrt{\cdots} \end{bmatrix} \qquad \frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$\begin{bmatrix} 0 & \sqrt{\frac{3}{2}-\frac{\sqrt5}{2}} \end{bmatrix} \qquad -\frac{1}{2}+\frac{\sqrt5}{2}$$

$$V = \begin{bmatrix} \sqrt{5-\sqrt5}/\sqrt{10} & -\sqrt{5+\sqrt5}/\sqrt{10} \\ \sqrt2/\sqrt{5-\sqrt5} & \sqrt2/\sqrt{5+\sqrt5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = U\Sigma V^T.$$

Problem 12 Practice Final:

$$A = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ 0 & d & 0 \end{bmatrix}$$ what can we say about the signs of the eigenvalues?

$$f_A(x) = \det \begin{bmatrix} -x & a & b \\ c & -x & 0 \\ 0 & d & -x \end{bmatrix} = -x\cdot(x^2) - c\cdot(-ax-bd) = -x^3 + acx + cbd.$$

We know that A has three distinct real eigenvalues.

$$f_A(x) = -(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$$

$$\det(A) = \lambda_1\lambda_2\lambda_3$$

A is similar to $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$$\mathrm{tr}(A) = \lambda_1+\lambda_2+\lambda_3$$

$$\det(A) = -c\cdot(-bd) = cbd > 0$$

$$\mathrm{tr}(A) = 0$$

Two negative, one positive, an the positive one is the largest.

Problem 7 Practice Final:

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad \begin{bmatrix} \cdots & \cdots \\ 6 & -9 \end{bmatrix}$ $\qquad \mathfrak{B} = \left\{ \begin{bmatrix} \cdot \\ 1 \end{bmatrix}, \begin{bmatrix} \cdot \\ 3 \end{bmatrix} \right\}$

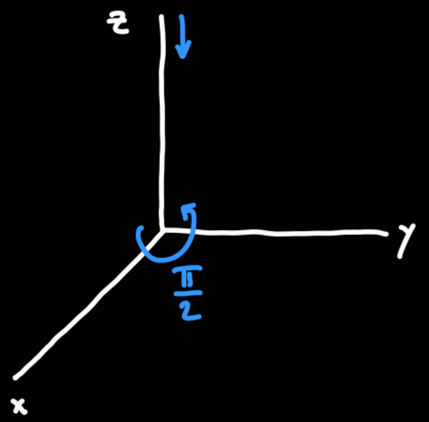
$$B = \left[\left[T \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]_{\mathfrak{B}} \quad \left[T \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right]_{\mathfrak{B}} \right] = \left[\begin{bmatrix} 6 \\ 3 \end{bmatrix}_{\mathfrak{B}} \quad \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathfrak{B}} \right] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\vec{x} = S\,[\vec{x}]_{\mathfrak{B}}$$

$$\overset{n\times m}{A} : \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad \overset{m\times n}{A^T} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\overset{m\times m}{A^TA} : \mathbb{R}^m \longrightarrow \mathbb{R}^m \qquad AA^T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Problem 3 Practice Final:



$$\left[\left[T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]_{\mathfrak{S}} \quad \left[T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right]_{\mathfrak{S}} \quad \left[T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]_{\mathfrak{S}} \right] =$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 1(e) Midterm 2:

There are real invertible 3x3 matrices A, S with $S^TAS = -A$.

False: $\det(S^TAS) = \det(-A) = (-1)^3 \cdot \det(A) = -\det(A)$

$$\| $$

$$\det(S^T)\det(A)\det(S)$$

$$\| $$

$$\det(S)^2 \cdot \det(A)$$

So: $\det(S)^2 = -1$. So: $\det(S) = 0$, but $\det(S) \neq 0$. Contradiction.

$\det(A) \neq 0$