Theorem 5.4.1.: $\quad \text{im}(A)^{\perp} = \ker(A^T).$

Given $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ an $n \times m$ matrix, it corresponds to a linear transformation $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ where:

$$\text{im}(A) = \text{span}(\vec{v}_1, \ldots, \vec{v}_m) \text{ is a linear subspace of } \mathbb{R}^n.$$

To compute the orthogonal complement of $\text{im}(A)$, we use the definition:

$$\text{im}(A)^{\perp} = \{ \vec{x} \text{ in } \mathbb{R}^n \mid \vec{v} \cdot \vec{x} = \vec{0} \text{ for all } \vec{v} \text{ in } \text{im}(A) \}.$$

Since $\text{im}(A) = \text{span}(\vec{v}_1, \ldots, \vec{v}_m)$ then $\vec{v} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$ for some $c_1, \ldots, c_m$ real numbers. Then:

$$\vec{v} \cdot \vec{x} = \vec{0} \text{ for all } \vec{v} \text{ in } \text{im}(A) \text{ implies that } \vec{v}_1 \cdot \vec{x} = 0, \ldots, \vec{v}_m \cdot \vec{x} = \vec{0}.$$

Also:

$$\vec{v}_1 \cdot \vec{x} = \vec{0}, \ldots, \vec{v}_m \cdot \vec{x} = 0 \text{ implies that } \vec{v} \cdot \vec{x} = (c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m) \cdot \vec{x} =$$

$$= c_1 \vec{v}_1 \cdot \vec{x} + \cdots + c_m \vec{v}_m \cdot \vec{x} = 0$$

for all $\vec{v}$ in $\text{im}(A)$.

Namely, the statement "$\vec{v} \cdot \vec{x} = \vec{0}$ for all $\vec{v}$ in $\text{im}(A)$" and the statement "$\vec{v}_1 \cdot \vec{x} = 0, \ldots, \vec{v}_m \cdot \vec{x} = 0$" are equivalent. Then:

$$\text{im}(A)^{\perp} = \{ \vec{x} \text{ in } \mathbb{R}^n \mid \vec{v} \cdot \vec{x} = \vec{0} \text{ for all } \vec{v} \text{ in } \text{im}(A) \}$$

$$\mathrm{im}(A)^{\perp} = \{\vec{x} \text{ in } \mathbb{R}^n \mid \vec{v}\cdot\vec{x} = \vec{0} \text{ for all } \vec{v} \text{ in } \mathrm{im}(A)\}$$

$$= \{\vec{x} \text{ in } \mathbb{R}^n \mid \vec{v}_1\cdot\vec{x} = \vec{0}, \dots, \vec{v}_m\cdot\vec{x} = \vec{0}\}$$

$$= \{\vec{x} \text{ in } \mathbb{R}^n \mid \vec{v}_1^{T}\vec{x} = \vec{0}, \dots, \vec{v}_m^{T}\vec{x} = \vec{0}\}$$

$$= \left\{\vec{x} \text{ in } \mathbb{R}^n \;\middle|\; \begin{bmatrix} - \vec{v}_1^{T} - \\ \vdots \\ - \vec{v}_m^{T} - \end{bmatrix}\vec{x} = \vec{0}\right\}$$

$$= \{\vec{x} \text{ in } \mathbb{R}^n \mid A^T\vec{x} = \vec{0}\} = \ker(A^T).$$

Since $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ yields $A^T = \begin{bmatrix} - \vec{v}_1^{T} - \\ \vdots \\ - \vec{v}_m^{T} - \end{bmatrix}$.

Remark: In general, we cannot expect $\mathrm{im}(A)^{\perp} = \ker(A)$ because they live in different vector spaces:

$$\begin{array}{ccc} A: \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \\ \cup\!\shortmid & & \cup\!\shortmid \\ \ker(A) & & \mathrm{im}(A) \end{array}$$

Given A an $n \times m$ matrix, its kernel is a subspace of $\mathbb{R}^m$, while its image is a subspace of $\mathbb{R}^n$. For example, consider the linear transformation given by the matrix $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$. This is a 1×2 matrix, giving a linear transformation $\mathbb{R}^2 \longrightarrow \mathbb{R}^1$. Since $\mathrm{im}(A) = \mathrm{span}(1,1) = \mathrm{span}(1) = \mathbb{R}^1$, then $\mathrm{im}(A)^{\perp} = \{0\}$, there is only one vector in $\mathbb{R}^1$ perpendicular to $\mathrm{im}(A)$. However, the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is in the kernel of A: $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1+1 = 0$ and thus $\mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$ is in the kernel of A. Now:

$$\ker(A) \text{ contains } \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \text{ which is not } \{0\} = \mathrm{im}(A)^{\perp},$$

and thus $\operatorname{im}(A)^\perp \neq \ker(A)$.

Problem 5.3.35.: Find an orthogonal transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that:

$$T\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Main idea!: An orthogonal transformation has matrix where the columns form an orthonormal basis.

Equivalently, an orthogonal transformation sends orthonormal basis to orthonormal basis.

$$\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & [T(\vec{v}_2)]_{\mathfrak{B}} & [T(\vec{v}_3)]_{\mathfrak{B}} \end{bmatrix} \qquad B = \begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix}$$

$$\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = T^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ in the standard basis the matrix associated to T^{-1} is:

$$A = \begin{bmatrix} * & * & 2/3 \\ * & * & 2/3 \\ * & * & 1/3 \end{bmatrix} \qquad \begin{matrix} -1/3 \\ 2/3 \\ -2/3 \end{matrix}$$ so T has A^T as matrix.

$\vec{c}_1 \cdot \vec{c}_1 = 1$, $\vec{c}_2 \cdot \vec{c}_2 = 1$, $\vec{c}_3 \cdot \vec{c}_3 = 1$, $\vec{c}_i \cdot \vec{c}_j = \vec{0}$ for $i \neq j$.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \hat{\imath} \cdot 1 - \hat{\jmath} \cdot 1 + \hat{k} \cdot 1 \rightsquigarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$