Gmail

Pablo S. Ocal <pablosanchezocal@gmail.com>

Re: Doubts - Algebra II Class

1 message

Matt Papanikolas <papanikolas@tamu.edu> Sat, Mar 11, 2017 at 9:22 PM
To: Pablo Sánchez Ocal <pablosanchezocal@gmail.com>

Pablo,

It might be useful to present some of the various isomorphisms in more detail. First we start with the $\mathbb{Q}$-vector space $A = \oplus_{i=1}^{\infty}\mathbb{Q}$, and then we note that we have $\mathbb{Q}$-vector space isomorphisms

$$A \cong V \oplus W \cong A \oplus A,$$

where $V = \oplus_{i\,\mathrm{odd}}\mathbb{Q}$ and $W = \oplus_{i\,\mathrm{even}}\mathbb{Q}$. The first isomorphism is fairly straightforward, and we will set

$$\mu : V \xrightarrow{\sim} A, \qquad \nu : W \xrightarrow{\sim} A,$$

to be the natural isomorphisms, which we use to define the second isomoprhism $\mu \oplus \nu : V \oplus W \xrightarrow{\sim} A \oplus A.$ The actual definitions of μ and ν won't matter -- we just need them to be isomorphisms.

Now by way of μ we have an isomorphism of rings

$$\tilde{\mu} : \mathrm{Hom}_{\mathbb{Q}}(A, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(V, V).$$

But let's work this out in more detail. We have

$$\tilde{\mu} : \mathrm{Hom}_{\mathbb{Q}}(A, A) \xrightarrow{\mu^*} \mathrm{Hom}_{\mathbb{Q}}(V, A) \xrightarrow{(\mu^{-1})_*} \mathrm{Hom}_{\mathbb{Q}}(V, V).$$

What this means is that if $f : A \to A$ is $\mathbb{Q}$-linear, then $\tilde{\mu}(f) : V \to V$ is defined by

$$\tilde{\mu}(f) = \mu^{-1} \circ f \circ \mu.$$

Ok, so now we let $R = \mathrm{Hom}_{\mathbb{Q}}(A, A)$, and we see that $\mathrm{Hom}_{\mathbb{Q}}(V, V)$ is a free left R-module via $\tilde{\mu}$ with basis $\{\mathrm{id}_V\}$. For $g \in R$ and $\alpha \in \mathrm{Hom}_{\mathbb{Q}}(V, V)$, under this R-operation we trace through the definitions and find

$$g \cdot \alpha = \mu^{-1} \circ g \circ \mu \circ \alpha.$$

And in particular $\mathrm{Hom}_{\mathbb{Q}}(V, V) \cong R$ as a left R-module. Similarly we can define

$$\tilde{\nu} : \mathrm{Hom}_{\mathbb{Q}}(A, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(W, W),$$

and $\mathrm{Hom}_{\mathbb{Q}}(W, W)$ also has an induced left R-module structure and is isomorphic to R as a left R-module.

On to the problem at hand, we have a decomposition

$$\begin{aligned} R = \mathrm{Hom}_{\mathbb{Q}}(A, A) &\cong \mathrm{Hom}_{\mathbb{Q}}(V \oplus W, A) \\ &\cong \mathrm{Hom}_{\mathbb{Q}}(V, A) \oplus \mathrm{Hom}_{\mathbb{Q}}(W, A) \\ &\cong \mathrm{Hom}_{\mathbb{Q}}(V, V) \oplus \mathrm{Hom}_{\mathbb{Q}}(W, W), \end{aligned}$$

where the map $\phi : \mathrm{Hom}_{\mathbb{Q}}(A, A) \to \mathrm{Hom}_{\mathbb{Q}}(V, V) \oplus \mathrm{Hom}_{\mathbb{Q}}(W, W)$ is a $\mathbb{Q}$-vector space isomorphism via the maps defined above. In particular, for $f : A \to A$,

$$\phi(f) = (\mu^{-1} \circ f \circ \mu, \nu^{-1} \circ g \circ \nu).$$

To show that ϕ is an isomorphism of left R-modules, we need to verify that for $g \in R$,

$$\phi(g \circ f) = g \cdot \phi(f),$$

but we note that

$$
\begin{aligned}
\phi(g \circ f) &= (\mu^{-1} \circ g \circ f \circ \mu, \nu^{-1} \circ g \circ f \circ \nu) \\
&= (g \cdot (\mu^{-1} \circ f \circ \mu), g \cdot (\nu^{-1} \circ f \circ \nu)) \\
&= g \cdot (\mu^{-1} \circ f \circ \mu, \nu^{-1} \circ f \circ \nu) \\
&= g \cdot \phi(f).
\end{aligned}
$$

Thus ϕ induces an isomorphism $R \cong R \oplus R$ of left R-modules.

I hope this is helpful. Enjoy the break.

Matt

On Fri, Mar 10, 2017 at 2:20 PM, Pablo Sánchez Ocal <pablosanchezocal@gmail.com> wrote:

> Dear Dr. Papanikolas,
>
> Today in class I asked you why:
>
> $\pi_i \circ g \circ f = g \circ \pi \circ f,$
>
> I just now tried to do it component-wise, but I found that there was no reason for them to be equal.
>
> Why must this happen?
>
> Thank you,
>
> Pablo Sánchez