(7) Roots of f : $\pm\sqrt{2\pm\sqrt{2}}$.

Let $\alpha = 2+\sqrt{2}$, $\beta = 2-\sqrt{2}$. Then

$E = \mathbb{Q}(\alpha, \beta, -\alpha, -\beta) \Rightarrow E/\mathbb{Q}$ is Galois

and $|\mathrm{Gal}(E/\mathbb{Q})| = 4$. So, $\mathrm{Gal}(E/\mathbb{Q}) \cong C_4$ or $C_2 \times C_2$.

Since $\alpha\beta = 2$, we have $\beta = \frac{2}{\alpha}$. Hence

$E = \mathbb{Q}(\sqrt{\alpha})$.

$$
\begin{tikzcd}
\text{Now,} \quad E = \mathbb{Q}(\alpha) = \mathbb{Q}(\beta). \arrow[d, dash, "2"'] \\
K := \mathbb{Q}(\sqrt{2}) = \mathbb{Q}(2+\sqrt{2}) = \mathbb{Q}(2-\sqrt{2}) \arrow[d, dash, "2"'] \\
\mathbb{Q}
\end{tikzcd}
$$

$\sqrt{\alpha}$ is algebraic over K w/

$f(x) := \mathrm{Irr}(\sqrt{\alpha}, K, x) = x^2 - \alpha = x^2 - (2+\sqrt{2})$.

$\sqrt{\beta}$ is a root of :

$f^{\sigma}(x) = \mathrm{Irr}(\sqrt{\beta}, K, x) = x^2 - \sigma(\alpha) = x^2 - (2-\sqrt{2})$
$= x^2 - \beta$.

where ~~$\sigma: K \to K$~~ $\sigma \in \mathrm{Aut}(K/\mathbb{Q})$

~~$a+b\sqrt{2} \mapsto a-b\sqrt{2}$.~~

and $\sigma(2+\sqrt{2}) = 2-\sqrt{2}$.

So, there is a unique $\tau \in Aut(E/\mathbb{Q})$ for which $\tau(\sqrt{\alpha}) = \sqrt{\beta}$ and $\tau|_K = \sigma$.

Now, $\tau(\sqrt{\alpha}) = \sqrt{\beta}$ and $\tau(2+\sqrt{2}) = 2-\sqrt{2}$.

$$\tau(\sqrt{\beta}) = \frac{\sqrt{2}}{\sqrt{\alpha}} = \frac{-\sqrt{2}}{\sqrt{\beta}} = -\sqrt{\alpha}.$$

Hence $|\tau| \notin \{1,2\}$. So, $|\tau| = 4$.

$\Rightarrow Gal(E/\mathbb{Q}) = \langle \tau \rangle$

$\cong \mathbb{Z}/4\mathbb{Z}$. ▨