August 2018

Sunday, 26 July 2020 19:00

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① - G finite group. Prove: $|\{(g,h)\in G\times G \mid gh=hg\}| = k\cdot|G|$ where k is the number of conjugacy classes in G.

Remark: This looks like an Orbit-Stabilizer Theorem type of problem.

$\mathcal{O}_g := \{gxg^{-1} \mid x\in G\}$ is the conjugacy class of $g\in G$.

$S = \{(g,h)\in G\times G \mid gh=hg\}$.

Consider $\phi_g: G \longrightarrow G$, $h \longmapsto ghg^{-1}$ the conjugation by g (induces a group action). Let $\mathcal{O} = \{\mathcal{O}_g \mid g\in G\}$ the set of conjugacy classes of G. The problem asks to prove: $|S| = |\mathcal{O}|\cdot|G|$.

To use the Orbit-Stabilizer Theorem, set: $F(\phi_g) := \{h\in G \mid \phi_g(h) = h\}$ the set of fixed points of ϕ_g. Now: $|Stab(g)| = |\{h\in G \mid \underbrace{ghg^{-1}}_{\text{group action}} = h\}| = |\{h\in G \mid \phi_g(h)=h\}| = |F(\phi_g)|$

So:

$|F(\phi_g)|\cdot|\mathcal{O}_g| = |G|$ by the Orbit-Stabilizer Theorem.

Moreover: $$|S| = \sum_{g\in G} |F(\phi_g)| = \sum_{\mathcal{O}_g\in\mathcal{O}}\sum_{h\in\mathcal{O}_g}|F(\phi_g)| = \sum_{\mathcal{O}_g\in\mathcal{O}} |\mathcal{O}_g|\cdot|F(\phi_g)| = \sum_{\mathcal{O}_g\in\mathcal{O}} |G| = |G|\cdot|\mathcal{O}|.$$

(third equality: $|F(\phi_g)|$ constant on conjugacy classes; fourth equality: Orbit-Stabilizer Theorem.)

③ - $R = \mathbb{C}[x,y]/(x^3,y^3)$.

(a) Find all prime ideals.

We know that there is a one-to-one correspondence:

$\{\text{prime ideals of } R/I\} \longleftrightarrow \{P/(x^3,y^3) \text{ where } P \text{ is a prime ideal of } R \text{ containing } (x^3,y^3)\}$.

Now if $(x^3,y^3)\subseteq P$ prime, then $x^3\in P$ so $x\in P$ since P prime.
$y^3\in P$ $\quad y\in P$

Then $(x,y)\subseteq P$.

But (x,y) is maximal, so $(x,y)=P$. Hence R has only one prime ideal $\frac{(x,y)}{(x^3,y^3)}$.

(b) Show R has unique maximal ideal.

Every maximal ideal is prime, and the unique prime ideal of R is $\frac{(x,y)}{(x^3,y^3)}$, so

Every maximal ideal is prime, and the unique prime ideal of R is $(x,y)/(x^3,y^3)$, so since every prime ideal must be contained in a maximal ideal, the unique prime ideal is also the unique maximal ideal.

(c) Find all units of R.

We use the following result: a ring with unit R is local iff the set of non-unit elements of R is an ideal of R.

By (b), we have that $\mathbb{C}[x,y]/(x^3,y^3)$ is local. Moreover, $\bar{x}$ is not a unit because it is a zero divisor: $\bar{x}\cdot\bar{x}^2 = 0$. If $\bar{u}\cdot\bar{x} = 1$ then $\bar{x}^2 = \bar{x}^2\cdot 1 = \bar{x}^2\cdot\bar{x}\cdot\bar{u} = \bar{0}\cdot\bar{u} = \bar{0}$, contradiction. Similarly, $\bar{y}$ is not a unit. Then, the set of non-units contains $\bar{x},\bar{y}$, and since it must be an ideal, it contains $(\bar{x},\bar{y})$. Moreover, since all elements of degree 0 are invertible, the ideal of non-units cannot contain polynomials with non-zero constant term. Hence the ideal of non-units is $(x,y)/(x^3,y^3)$, and the units are $\mathbb{C}[x,y]/(x^3,y^3) - (x,y)/(x^3,y^3)$. That is, any polynomial with a non-zero constant term is a unit. details

9 - Find the Galois group of the splitting field of x^4-3 over $\mathbb{Q}[i]$.

Note: $x^4-3 = (x-\sqrt[4]{3})(x+\sqrt[4]{3})(x-i\sqrt[4]{3})(x+i\sqrt[4]{3})$, and irreducible over $\mathbb{Q}[i]$ since $\sqrt{3}, \sqrt[4]{3} \notin \mathbb{Q}[i]$.
Then the splitting field of x^4-3 over $\mathbb{Q}[i]$ is $\mathbb{Q}[i](\sqrt[4]{3})$ since all four roots are there, and: $[\mathbb{Q}[i](\sqrt[4]{3}):\mathbb{Q}[i]] = 4$. Let $G = Gal(\mathbb{Q}[i](\sqrt[4]{3})/\mathbb{Q}[i])$, we know $|G| = 4$.

Since any $\sigma \in G$ permutes the roots of x^4-3, and we must have $\sigma(i) = i$, it is enough to determine $\sigma(\sqrt[4]{3})$ to fully determine σ, and we have at most (*) four possibilities for it: details

$\sigma_0(\sqrt[4]{3}) = \sqrt[4]{9}$: $\sqrt[4]{3} \mapsto \sqrt[4]{3}$, $-\sqrt[4]{3} \mapsto -\sqrt[4]{3}$, $i\sqrt[4]{3} \mapsto i\sqrt[4]{3}$, $-i\sqrt[4]{3} \mapsto -i\sqrt[4]{3}$

$\sigma_0 = id.$

$\sigma_1(\sqrt[4]{3}) = \sqrt[4]{9}$: $\sqrt[4]{3} \mapsto -\sqrt[4]{3}$, $-\sqrt[4]{3} \mapsto \sqrt[4]{3}$, $i\sqrt[4]{3} \mapsto -i\sqrt[4]{3}$, $-i\sqrt[4]{3} \mapsto i\sqrt[4]{3}$

$|\sigma_1| = 2$

$\sigma_2(\sqrt[4]{3}) = \sqrt[4]{9}$: $\sqrt[4]{3} \mapsto i\sqrt[4]{3}$

$\sigma_3(\sqrt[4]{3}) = \sqrt[4]{9}$: $\sqrt[4]{3} \mapsto -i\sqrt[4]{3}$

$\sigma_2(\sqrt[4]{3}) = \sqrt[4]{3}$:

$|\sigma_2| = 4$

$$\sqrt[4]{3} \longmapsto i\sqrt[4]{3}$$
$$-\sqrt[4]{3} \longmapsto -i\sqrt[4]{3}$$
$$i\sqrt[4]{3} \longmapsto -\sqrt[4]{3}$$
$$-i\sqrt[4]{3} \longmapsto \sqrt[4]{3}$$

,

$\sigma_3(\sqrt[4]{3}) = \sqrt[4]{3}$:

$|\sigma_3| = 4$

$$\sqrt[4]{3} \longmapsto -i\sqrt[4]{3}$$
$$-\sqrt[4]{3} \longmapsto i\sqrt[4]{3}$$
$$i\sqrt[4]{3} \longmapsto \sqrt[4]{3}$$
$$-i\sqrt[4]{3} \longmapsto -\sqrt[4]{3}$$

And thus $G \cong \mathbb{Z}_4$.

⊛ Since we have at most four options for σ, and $|G| = 4$, all of those options must happen.