Introduction to ODEs and PDEs - Final Exam

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Exercise 1

Formulate the vector field rectification theorem. Formulate and derive from this theorem the dependence on parameters theorem.

Vector Field Rectification Theorem: Any vector field of class $\mathcal{C}^r$ in a neighborhood of any its non-singular point x_0 can be reduced to a constant field by applying a diffeomorphism of class $\mathcal{C}^r$.

Dependence on Parameters Theorem: Let the vector field $F(x, \mu)$ (where μ belongs to an open domain of a space $\mathbb{R}^m$) be of class $\mathcal{C}^r$. Let also $F(x_0, \mu_0) \neq 0$. Then the unique solution $x(t, t_0, x, x_0, \mu)$ of the initial value problem $dx/dt = F(x, \mu)$, $x(t_0) = x_0$ depends differentiably of class $\mathcal{C}^r$ on (t, t_0, x, x_0, μ) for sufficiently small $|t-t_0|$, $|x - x_0|$, $|\mu - \mu_0|$.

We will first remark that the system $x'(t) = F$ with $x(t_0) = x_0$ for $F \in \mathbb{R}$ a constant has a unique solution (easily obtained by integrating and imposing the initial condition), namely $x(t) = Ft + x_0 - Ft_0$. Using a rectifying diffeomorphism, we will reduce us to this case, where we already know the result.

Let the vector field $F(x, \mu)$ (where μ belongs to an open domain of a space $\mathbb{R}^m$) be of class $\mathcal{C}^r$. Let also $F(x_0, \mu_0) = 0$. We prove that the (unique) solution $x(t, t_0, x, x_0, \mu)$ of the initial value problem $x'(t) = F(x, \mu)$ and $x(t_0) = x_0$ depends differentiably of class $\mathcal{C}^r$ on (t, t_0, x, x_0, μ) for sufficiently small $|t - t_0|$, $|x - x_0|$, $|\mu - \mu_0|$.

For this, we note that we may do the standard supposition (already implicit in the statement of the Dependence on Parameters Theorem) that F does not have any dependence on t, and we now transform the system to consider μ as part of the initial conditions, by rewriting $z(t) = \mu$, which gives us:

$$\begin{cases} x'(t) = F(x, \mu) \\ x(t_0) = x_0 \end{cases} \qquad \text{becomes} \qquad \begin{cases} x'(t) = F(x, z) \\ z'(t) = 0 \\ x(t_0) = x_0 \\ z(t_0) = \mu \end{cases}$$

so we only need to prove the dependence on time and initial conditions for an autonomous system $x'(t) = F(x)$ and $x(t_0) = x_0$.

1. If $F(x)$ has x_0 as a singular point, then we add $s = t$ with $s'(t) = 1$ and $s(t_0) = t_0$ to our system, resulting in:

$$\begin{cases} x'(t) = F(x) \\ s'(t) = 1 \\ x(t_0) = x_0 \\ s(t_0) = t_0 \end{cases}$$

   which is a non-autonomous system since the derivative of $s(t)$ is always 1 (hence never zero). We are thus reduced to the following case.

2. If $F(x)$ does not have x_0 as a singular point, then the rectification theorem gives us a local diffeomorphism G of class $\mathcal{C}^r$ with $G(x(t)) = y(t)$ and transforms our system into $y'(t) = DG(G^{-1}(y))F(G^{-1}(y))$, constant, and $y(t_0) = G(x(t_0)) = y_0$. This system is of the form we remarked above, thus we can indeed find $y(t)$ a unique solution. This means that $x(t) = G^{-1}(y(t))$ is the unique solution of our original system, valid only near a neighborhood of x_0 since G is only local.

Notice that the solution $y(t)$ arises from a system that has dependence $\mathcal{C}^\infty$ with respect to time and initial conditions. This means that the solution $x(t) = G^{-1}(y(t))$ depends differentiably $\mathcal{C}^r$ with respect to time and initial conditions, since G is only a local diffeomorphism of class $\mathcal{C}^r$ and we may lose smoothness. This locality guarantees the smoothness for sufficiently small $|t - t_0|$, $|x - x_0|$, which is what we wanted to prove.

Exercise 2

1. Can there exist a non trivial (that is non identically zero) vibration of an infinite string such that a non zero segment $[a, b] \subset \mathbb{R}$ does not move for any $t > 0$?

   Consider the initial value problem:

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\ u = g \quad \text{in } \mathbb{R} \times \{0\} \\ u_t = -g' \quad \text{in } \mathbb{R} \times \{0\} \end{cases}$$

   where $g \in \mathcal{C}^2$.

   The condition over g guarantees that we can applying d'Alembert's formula and indeed obtain a solution of the initial value problem. This solution is:

$$\begin{aligned} u(t, x) &= \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} -g'(y)dy \\ &= \frac{1}{2}[g(x+t) + g(x-t)] + \frac{-1}{2}g'(y)\Big|_{x-t}^{x+t} \\ &= \frac{1}{2}[g(x+t) + g(x-t)] - \frac{1}{2}g(x+t) + \frac{1}{2}g(x-t) \\ &= g(x-t). \end{aligned}$$

   This means that the solution is a wave that only propagates towards the positive x. Hence taking g not zero with compact support, say $\sup(g) = [0, 1]$, the solution will never touch $x < 0$ (it will never move them) and is not identically zero. Thus any segment inside $(-\infty, 0)$, say $[-2, -1]$, is never moved for all $t > 0$. This construction proves that the answer to the question is affirmative.

2. Can there exist a non trivial (that is non identically zero) vibration of an infinite string such that a non zero segment $[a, b] \subset \mathbb{R}$ does not move for any $-\infty < t < \infty$?

   If we accept that $u(t, x) = x$ is a function that is not moving a non zero segment (since we indeed have that "$u(t, x) = x = u(s, x)$ for all $t, s \in \mathbb{R}$ and all $x \in [-1, 1]$" is an apparently valid interpretation of "a non zero segment $[-1, 1] \subset \mathbb{R}$ does not move for any $-\infty < t < \infty$"), the answer to this question is affirmative: $u(t, x) = x$ is a non trivial vibration of an infinite string such that a non zero segment $[-1, 1] \subset \mathbb{R}$ does not move for any $-\infty < t < \infty$.

   However, we would like to thread more carefully. Notice that constant non zero solutions are not explicitly said to be trivial, but we interpret that they should be. Similarly solutions of the form $u(t, x) = x$ and $u(t, x) = t$ are not identically zero, but they are abusing that we have two derivatives and we interpret that they are trivial. For these reasons, we now interpret what we mean by "a non zero segment $[a, b] \subset \mathbb{R}$ does not move for any $-\infty < t < \infty$". In the following, this will mean that $u(t, x) = u(s, y)$ for all $t, s \in \mathbb{R}$ and all $x, y \in [a, b]$.

Under the interpretation exposed above, the answer to this question is negative. To prove it, we will use the fact that the general solution of the one dimensional wave equation is of the form $u(t,x) = F(x+t) + G(x-t)$ for some appropriate functions F and G [1, Remark (p. 68)]. Note that F represents a wave that travels from ∞ to $-\infty$, while G represents a wave that travels from $-\infty$ to ∞.

Suppose that there is such a function u, non trivial vibration of an infinite string, that has a non zero segment that never moves for $t \in \mathbb{R}$. Note first that by translation we can safely assume that $[-1,1] \subset \mathbb{R}$ is the segment that u doesn't move (we are interested in having 0 inside the segment for simplification purposes). We will achieve a contradiction by looking at how F and G behave locally.

(a) If G and F are both constant, this clearly defines u constant (and not necessarily zero) a solution of the wave equation. However, one may say that this should be considered inside the trivial case.

(b) If G is constant but F is not constant, since F is a wave equation traveling along the whole string and we are considering that the time may go backwards (notice how this is a key difference with the case considered in the first part of this Exercise) this means that the part of F that is not constant will eventually reach our segment $[-1,1]$, perturbing it and contradicting the hypothesis.

(c) If G and F are both not constant, consider $-s \in \mathbb{R}$ a point where G is not constant in a neighborhood, say $G(-s)$ is increasing as the position increases. This not may always be the case, the other options of the cases of G decreasing as the position increases, G increasing as the position decreases and G decreasing as the position decreases are all analogous to the one we treat here.

  i. If $F(s)$ is constant as the position increases, this immediately yields a contradiction since $G(-s) < G(\epsilon - s)$ implies $u(s,0) = F(s) + G(-s) < F(\epsilon + s) + G(\epsilon - s) = u(s,\epsilon)$ for $\epsilon \in (0,1]$, and then u perturbs $[-1,1]$. Alternatively for $r < s$, note that G increasing means $G(-r) > G(-s)$ and thus $u(s,0) = F(s)+G(-s) < F(r)+G(-s) = u(r,0)$ and u perturbs $[-1,1]$.

  ii. If $F(s)$ increases as the position increases, we consider $x = 0$ and $t = s$, the value of $u(0,s)$ is arbitrary since we only claim that u doesn't perturb $[-1,1]$, so without loss of generality we may assume $u(s,0) = F(s) + G(-s) = 0$. Then considering $x = \epsilon \in (0,1)$ and $t = s - \epsilon$, we find that:

$$\begin{aligned} u(s-\epsilon,\epsilon) &= F(\epsilon + s - \epsilon) + G(\epsilon - s + \epsilon) = F(s) + G(2\epsilon - s) \\ &= G(2\epsilon - s) - G(-s) > 0 \end{aligned}$$

  since G is increasing. Since u perturbs $[-1,1]$, this is again a contradiction.

  iii. If $F(s)$ decreases as the position increases, we consider $x = 0$ and $t = s$, then comparing it with $x = 0$ and $t = r < s$, we find that:

$$u(s,0) = F(s) + G(-s) < F(r) + G(-r) = u(r,0)$$

since F decreasing means $F(r) > F(s)$ and G increasing means $G(-r) > G(-s)$. Since u perturbs $[-1, 1]$, this is again a contradiction.

Exercise 3

1. Using Fourier transform, prove that the Laplace operator has no eigenfunctions in $L_2(\mathbb{R}^n)$, that is, if $-\Delta u = \lambda u$ for some $\lambda \in \mathbb{C}$ and $\int_{\mathbb{R}^n} |u(x)|^2 dx < \infty$, then $u = 0$.

   Since we want the solution to be $u \in L_2(\mathbb{R}^n)$, assume such a solution exists (we have $n \geq 2$ since otherwise this is not a partial differential equation, its solution is trivial and it does not lie in $L_2(\mathbb{R})$). We can then take the Fourier transform of the equation $-\Delta u = \lambda u$ and, in virtue of [1, Theorem 2 (p. 184)] and being careful with the multi-index notation, it becomes $|y|^2 \tilde{u} = \lambda \tilde{u}$. Assume $\tilde{u} \neq 0$ at some point $y_0 \in \mathbb{R}^n$, since $u \in L_2(\mathbb{R}^n)$ and the Fourier transform behaves well, we have that $\tilde{u}$ is continuous and thus $\tilde{u} \neq 0$ in an open neighborhood $\Omega \subseteq \mathbb{R}^n$ of y_0. Now in this Ω we have that $\tilde{u}$ satisfies the equation above and is non zero, hence we can divide by $\tilde{u}|_\Omega$ and find that $|y|^2 = \lambda$. This means that $\Omega \subseteq \{y \in \mathbb{R}^n : |y|^2 = \lambda\}$ which is a 1-dimensional graph in $\mathbb{R}^n$ with $n \geq 2$, hence it is not open. Moreover, the subsets of $\{y \in \mathbb{R}^n : |y|^2 = \lambda\}$ are of dimension 1 or 0, meaning that Ω is not open, a contradiction.

   Hence, there are no eigenfunctions of the Laplace operator in $L_2(\mathbb{R}^n)$.

2. Construct a non zero solution of $-\Delta u = \lambda u$ in $\mathbb{R}^n$.

   Given $\lambda \in \mathbb{C}$, we denote by $\sqrt{\lambda/n}$ the principal square root of the complex number $\lambda/n \in \mathbb{C}$ and:

$$\cos\left(\sqrt{\frac{\lambda}{n}} x_j\right) = \frac{1}{2}\left(e^{i\sqrt{\frac{\lambda}{n}}x_j} + e^{-i\sqrt{\frac{\lambda}{n}}x_j}\right)$$

   for x_j a variable and $j = 1, \ldots, n$. Note that:

$$\begin{aligned}
\frac{d}{dx_j}\left(\cos\left(\sqrt{\frac{\lambda}{n}} x_j\right)\right) &= -\sqrt{\frac{\lambda}{n}} \sin\left(\sqrt{\frac{\lambda}{n}} x_j\right) = -\sqrt{\frac{\lambda}{n}} \frac{1}{2i}\left(e^{i\sqrt{\frac{\lambda}{n}}x_j} - e^{-i\sqrt{\frac{\lambda}{n}}x_j}\right) \\
\frac{d^2}{dx_j^2}\left(\cos\left(\sqrt{\frac{\lambda}{n}} x_j\right)\right) &= -\frac{\lambda}{n} \cos\left(\sqrt{\frac{\lambda}{n}} x_j\right) = -\frac{\lambda}{n} \frac{1}{2}\left(e^{i\sqrt{\frac{\lambda}{n}}x_j} + e^{-i\sqrt{\frac{\lambda}{n}}x_j}\right)
\end{aligned}$$

   meaning that if we define:

$$u(x_1, \ldots, x_n) = \prod_{j=1}^n \cos\left(\sqrt{\frac{\lambda}{n}} x_j\right)$$

   we obtain that:

$$\begin{aligned}
\Delta u &= \sum_{k=1}^n \frac{d^2}{dx_k} u = \sum_{k=1}^n \frac{d^2}{dx_k} \prod_{j=1}^n \cos\left(\sqrt{\frac{\lambda}{n}} x_j\right) \\
&= -\sum_{k=1}^n \frac{\lambda}{n} \prod_{j=1}^n \cos\left(\sqrt{\frac{\lambda}{n}} x_j\right) = -\lambda \prod_{j=1}^n \cos\left(\sqrt{\frac{\lambda}{n}} x_j\right) = -\lambda u
\end{aligned}$$

   and hence $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ satisfies that $-\Delta u = \lambda u$, as desired.

Exercise 4

Let $u(t,x)$ be a bounded solution of the one dimensional heat equation $u_t = u_{xx}$, defined for all $t > 0$ and satisfying the initial condition $u|_{t=0} = \phi(x)$ with $\phi(x)$ being a bounded continuous function on $\mathbb{R}$. Knowing that $\lim_{x\to\infty}\phi(x) = b$ and $\lim_{x\to-\infty}\phi(x) = c$, describe how $u(t,x)$ behaves when t goes to infinity.

For this, we first write $u(t,x)$ using the fundamental solution of the heat equation, obtaining:

$$u(t,x) = \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy.$$

A preliminary examination of $u(t,x)$ appears to hint that the desired limit is $(c+b)/2$.

Given $\epsilon > 0$, we will now prove that the distance between $u(t,x)$ and $(c+b)/2$ when t goes to infinity is less than ϵ. Since this will hold for every ϵ, we will have proven that:

$$\lim_{t\to\infty}\left|u(t,x) - \frac{c+b}{2}\right| < \epsilon \Longrightarrow \lim_{t\to\infty}\left|u(t,x) - \frac{c+b}{2}\right| = 0 \Longrightarrow \lim_{t\to\infty} u(t,x) = \frac{c+b}{2},$$

which is the desired result.

Notice that since $\lim_{y\to\infty}\phi(y) = b$, there exists $N_b \in \mathbb{N}$ such that if $y \geq N_b$ then $|\phi(y) - b| < \epsilon$, and analogously since $\lim_{y\to-\infty}\phi(y) = c$ there exists $N_c \in \mathbb{N}$ such that if $y \leq -N_c$ then $|\phi(y) - c| < \epsilon$. This will allow us to divide $u(t,x)$ in three separate integrals:

$$\begin{aligned}
\lim_{t\to\infty}\left|u(t,x) - \frac{c+b}{2}\right| &= \lim_{t\to\infty}\left|\frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy - \frac{c+b}{2}\right| \\
&= \lim_{t\to\infty}\left|\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy + \frac{1}{2\sqrt{\pi t}}\int_{-N_c}^{N_b} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy\right. \\
&\left. + \frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy - \frac{c+b}{2}\right| \\
&\leq \lim_{t\to\infty}\left|\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy + \frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy\right. \\
&\left. - \frac{c+b}{2}\right|
\end{aligned}$$

since the middle term was the integral of a continuous function over a compact set, with $\lim_{t\to\infty} e^{\frac{-(x-y)^2}{4t}}\phi(y) = \phi(y)$ which integrated is then bounded by a constant (notice how this immediately shows that what we are integrating is under the hypothesis of Lebesgue's Dominated Convergence Theorem and hence is uniformly convergent and thus we can indeed permute integral and limit), multiplied by a function tending to zero when t tends to infinity, meaning that the whole middle term tends to zero when t tends to infinity. This is why we write less than or equal, since we (secretly) applied the triangle inequality.

Since we are dealing with the integral of exponential with negative squared exponents, it will be useful to consider the error function. We recall that $Err(x) = \frac{2}{\pi}\int_0^x e^{-t^2}dt$ and that $\int e^{ay^2}dy = \frac{\sqrt{\pi}}{2\sqrt{a}}Err(\sqrt{a}y)$ for any constant $a \in \mathbb{R}$, which means that $\int e^{a(x-y)^2}dy = \frac{-\sqrt{\pi}}{2\sqrt{a}}Err(\sqrt{a}(x-y))$, and both previous integrals are indefinite. With the above, notice that:

$$\frac{1}{2\sqrt{\pi t}}\int e^{\frac{-(x-y)^2}{4t}}dy = \frac{-1}{2}Err\left(\frac{x-y}{2\sqrt{t}}\right).$$

Moreover, notice that $\lim_{x\to\infty} Err(x) = 1$, $Err(0) = 0$ and $\lim_{x\to-\infty} Err(x) = -1$. In particular, this means that:

$$\begin{aligned}\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}dy &= \frac{-1}{2}Err\left(\frac{x-y}{2\sqrt{t}}\right)\Bigg|_{-\infty}^{-N_c} \\ &= \frac{-1}{2}Err\left(\frac{x+N_c}{2\sqrt{t}}\right) + \frac{1}{2}\lim_{y\to\infty} Err\left(\frac{x+y}{2\sqrt{t}}\right) = \frac{1}{2}\end{aligned}$$

where the first term is clearly evaluated at zero and the second at infinity since y is stronger than $\sqrt{t}$, so the depedence of $t \to \infty$ is not relevant for the computation. Analogously:

$$\begin{aligned}\frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}dy &= \frac{-1}{2}Err\left(\frac{x-y}{2\sqrt{t}}\right)\Bigg|_{N_b}^{\infty} \\ &= \frac{-1}{2}\lim_{y\to\infty} Err\left(\frac{x-y}{2\sqrt{t}}\right) + \frac{1}{2}Err\left(\frac{x-N_b}{2\sqrt{t}}\right) = \frac{1}{2}\end{aligned}$$

where the second term is clearly evaluated at zero and the first at negative infinity since y is stronger than $\sqrt{t}$, so the depedence of $t \to \infty$ is not relevant for the computation, thus canceling out the negative sign it carries.

Remark now that since c and b are constants, we can put them inside the integrals, and using the above, rewrite:

$$\begin{aligned}\lim_{t\to\infty}\left|u(t,x) - \frac{c+b}{2}\right| &\le \lim_{t\to\infty}\Bigg|\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy - \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}cdy \\ &+ \frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}\phi(y)dy - \frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}bdy\Bigg| \\ &= \lim_{t\to\infty}\Bigg|\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}(\phi(y)-c)dy \\ &+ \frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}(\phi(y)-b)dy\Bigg| \\ &\le \lim_{t\to\infty}\left|\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}(\phi(y)-c)dy\right| \\ &+ \lim_{t\to\infty}\left|\frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}(\phi(y)-b)dy\right|\end{aligned}$$

and we finally see:

$$\begin{aligned}
\lim_{t\to\infty}\left|u(t,x)-\frac{c+b}{2}\right| &\le \lim_{t\to\infty}\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}|\phi(y)-c|dy \\
&+\lim_{t\to\infty}\frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}|\phi(y)-b|dy \\
&<\lim_{t\to\infty}\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{-N_c} e^{\frac{-(x-y)^2}{4t}}\epsilon dy \\
&+\lim_{t\to\infty}\frac{1}{2\sqrt{\pi t}}\int_{N_b}^{\infty} e^{\frac{-(x-y)^2}{4t}}\epsilon dy \\
&=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

This proves what we wanted.

Exercise 5

1. Find the characteristics of the Euler equation $u_t + uu_x = 0$.

   Applying the standard formulas, we obtain that reparametrizing in terms of s for the time and y for the space:

   $$\begin{aligned} \frac{dt}{ds} &= 1 \\ \frac{dx}{ds} &= u(s,y) \\ \frac{du}{ds} &= 0 \end{aligned}$$

   are the characteristics of the Euler equation.

2. Suggest an initial condition for this equation and solve the resulting Cauchy problem by the method of characteristics.

   As a side note, we would like to consider the standard initial condition $u(0,x) = f(x)$. However, when we apply the method of characteristics, we find that we need to know extra information about f, namely if $x = f(s)t + s$ can be solved to find $s(t,x)$, which is not always true for a general f.

   We will then consider the simple initial condition $u(0,x) = ax$ where $a \in \mathbb{R}$ is a constant. This adds to the characteristics of the Euler equation the initial conditions to be satisfied:

   $$\begin{aligned} &\frac{dt}{ds} = 1, && t(0,y) = 0 \\ &\frac{dx}{ds} = u(s,y), && x(0,y) = y \\ &\frac{du}{ds} = 0, && u(0,y) = ay \end{aligned}$$

   are the characteristics of the Euler equation with the initial conditions. The first and last equations can be immediately solved, yielding $t(s,y) = t(0,y) + s = s$ and $u(s,y) = u(0,y) = ay$. Now using the third we can substitute in the second, obtaining $dx/ds = ay$, which can now be easily solved yielding $x(s,y) = ays + x(0,y) = ays + y$. Since we have $t(s,y) = s$ and $x(s,y) = ays + y$, we can solve for $s(t,x)$ and $y(t,x)$, the first equation yielding $s(t,x) = t$ and substituting this into the second we obtain $x = aty(t,x) + y(t,x) = (at+1)y(t,x)$ hence $y(t,x) = x/(at+1)$. This means that we can substitute in $u(s,y) = ay$ obtaining:

   $$u(t,x) = u(s(t,x), y(t,x)) = as(t,x) = \frac{ax}{at+1}$$

   which is indeed the desired solution since it can be immediately checked that $u(0,x) = ax$ and:

   $$\frac{\partial u}{\partial t} = \frac{-a^2 x}{(at+1)^2}, \quad \frac{\partial u}{\partial x} = \frac{a}{at+1} \Longrightarrow u_t + uu_x = \frac{-a^2 x}{(at+1)^2} + \frac{ax}{at+1}\frac{a}{at+1} = 0.$$

References

[1] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.