Introduction to ODEs and PDEs - Homework 1

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Exercise 1

We solve by successive Picard iterations the initial value problem:

$$\begin{cases} x'(t) = x \\ x(0) = 1 \end{cases}$$

beginning with the function $x_0(t) = 1$. Now:

$$\begin{aligned} x_1(t) &= 1 + \int_0^t ds = 1 + t \\ x_2(t) &= 1 + \int_0^t (1+s)ds = 1 + t + \frac{t^2}{2} \\ x_3(t) &= 1 + \int_0^t (1 + s + \frac{s^2}{2})ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \end{aligned}$$

so we now prove by induction that the general form is $x_n(t) = \sum_{k=0}^n t^k/k!$. We already have $n = 0, 1, 2, 3$, so assuming true for any other n, we compute:

$$\begin{aligned} x_{n+1}(t) &= 1 + \int_0^t x_n(s)ds = 1 + \int_0^t x_{n-1}(s)ds + \int_0^t \frac{s^n}{n!}ds \\ &= x_n(t) + \left. \frac{s^{n+1}}{n!(n+1)} \right|_0^t = 1 + t + \cdots + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!} \end{aligned}$$

exactly what we desired. Thus taking limits:

$$x(t) = \lim_{n\to\infty} x_n(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

Exercise 2

We show that the fields arising from turning non-autonomous systems into autonomous ones are always non-singular at all points.

Consider $x'(t) = F(t, x)$ a non-autonomous system. Rewriting $s = t$ we obtain the system:

$$\begin{cases} s'(t) = 1 \\ x'(t) = F(s, x) \end{cases}$$

so rewriting $y = (s, x)$ and $G(y) = (1, F(y))$ we obtain the equivalent system $y'(t) = G(y)$. Since the first component of $G(y)$ is always a 1, hence non-zero, we always have $G(y) \neq 0$, hence the new system is always non-singular.

Exercise 3

Deduce the existence and uniqueness theorem from the theorem on rectifying a vector field near a non-singular point. For this, we will first remark that the system $x'(t) = F$ with $x(t_0) = x_0$ for $F \in \mathbb{R}$ a constant has a unique solution (easily obtained by integrating and imposing the initial condition), namely $x(t) = Ft + x_0 - Ft_0$. Using a rectifying diffeomorphism, we will reduce us to this case, where we already know the result.

Let $\Omega \subset \mathbb{R}^n$ an open domain, $(a, b) \subset \mathbb{R}$ open, $F(t, x)$ and $D_x F(t, x)$ continuous on $(a, b) \times \Omega$. Given any $(t_0, x_0) \in (a, b) \times \Omega$, we want to see that there exists a unique solution $x(t)$ of the initial value problem $x'(t) = F(t, x)$ and $x(t_0) = x_0$ defined in a neighborhood of t_0.

For this, we first transform the system into an autonomous one. Since we can always do this as shown in the Exercise above, we may assume that F already has no dependence on t, that is, we want to solve $x'(t) = F(x)$ with $x(t_0) = x_0$.

1. If $F(x)$ has x_0 as a singular point, then we add $s = t$ with $s'(t) = 1$ and $s(t_0) = t_0$ to our system, resulting in:

$$\begin{cases} x'(t) = F(x) \\ s'(t) = 1 \\ x(t_0) = x_0 \\ s(t_0) = t_0 \end{cases}$$

   which as noted in Exercise 2 is a non-autonomous system. We are thus reduced to the following case.

2. If $F(x)$ does not have x_0 as a singular point, then the rectification theorem gives us a local diffeomorphism G of class $\mathcal{C}^r$ with $G(x(t)) = y(t)$ and transforms our system into $y'(t) = DG(G^{-1}(y))F(G^{-1}(y))$, constant, and $y(t_0) = G(x(t_0)) = y_0$. This system is of the form we remarked above, thus we can indeed find $y(t)$ a unique solution. This means that $x(t) = G^{-1}(y(t))$ is the unique solution of our original system, valid only near a neighborhood of x_0 since G is only local. This proves what we wanted.

Exercise 4

We prove the dependence on parameters theorem using the rectification theorem. Again, we keep in mind the system remarked in Exercise 3.

Let the vector field $F(x, \mu)$ (where μ belongs to an open domain of a space $\mathbb{R}^m$) be of class $\mathcal{C}^r$. Let also $F(x_0, \mu_0) = 0$. We prove that the (unique) solution $x(t, t_0, x, x_0, \mu)$ of the initial value problem $x'(t) = F(x, \mu)$ and $x(t_0) = x_0$ depends differentiably of class $\mathcal{C}^r$ on (t, t_0, x, x_0, μ) for sufficiently small $|t - t_0|$, $|x - x_0|$, $|\mu - \mu_0|$.

For this, we note that we may again suppose that F does not have any dependence on t, and we now transform the system to consider μ as part of the initial conditions, by rewriting $z(t) = \mu$, which gives us:

$$\begin{cases} x'(t) = F(x, t) \\ x(t_0) = x_0 \end{cases} \qquad \text{becomes} \qquad \begin{cases} x'(t) = F(x, z) \\ z'(t) = 0 \\ x(t_0) = x_0 \\ z(t_0) = \mu \end{cases}$$

so we only need to prove the dependence on time and initial conditions for an autonomous system $x'(t) = F(x)$ and $x(t_0) = x_0$.

By the reasoned in Exercise 3 and in that notation, given an autonomous system $x'(t) = F(x)$ and $x(t_0) = x_0$, we notice that the solution $y(t)$ arises from a system that has dependence $\mathcal{C}^\infty$ with respect to time and initial conditions. This means that the solution $x(t) = G^{-1}(y(t))$ depends differentiably $\mathcal{C}^r$ with respect to time and initial conditions, since G is only a local diffeomorphism of class $\mathcal{C}^r$ and we may lose smoothness. This locality guarantees the smoothness for sufficiently small $|t - t_0|$, $|x - x_0|$, which is what we wanted to prove.

Exercise 5

A global rectifying theorem does not hold. Consider the phase curves given (which have a top asymptote going from left to right, a bottom asymptote going from right to left and curves in the middle that begin on the right, go to the center and then return to the right) and suppose we have a diffeomorphism that smoothly transforms it into straight curves facing the same direction. This is the same thing as saying that we can smoothly bend the curves above to make the ones in the middle of the two asymptotes straight.

What happens to the asymptote above the curves is that, when we bend the middle curves to straighten them so that they come from the bottom and end at the top, bending the top part of the middle curves forces the top asymptote to smoothly move to the left, changing its position so that it comes from the bottom and ends at the top.

In a similar way, what happens to the asymptote below the curves is that, when we bend the middle curves to straighten them so that they come from the bottom and end at the top, bending the bottom part of the middle curves forces the bottom asymptote to smoothly move to the left, changing its position so that it comes from the bottom and ends at the top.

Notice that since both are asymptotes, none of the curves are really touching them, so they don't lie among them, they lie "aside" on the left. Hence they cannot be separated by any curves (since the movements we made to the asymptotes were symmetrical and did not involve each other). This means that by symmetry the diffeomorphism should map both asymptotes to the same curve in the final diagram. However, this is clearly a contradiction with this transformation being a diffeomorphism, which needs to be bijective. Hence, a global rectifying theorem does not hold, as we claimed.

Exercise 6

An initial value problem for an ODE that has no solution is:

$$\begin{cases} x'(t) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases} \\ x(1) = 1 \end{cases}$$

since to satisfy $x'(t) = 1$ and $x(1) = 1$ we would need $x(t) = t$, which does not satisfy the rest of the conditions. We have no solution, as desired.

Exercise 7

A boundary value problem for an ODE that has no solution is:

$$\begin{cases} x'(t) = 1 \\ y'(t) = -1 \\ x(0) = 0 \\ x(1) = 0 \end{cases}$$

since a general solution of the first equation is $x(t) = t + C$ for some constant $C \in \mathbb{R}$, but when we try to determine this constant we obtain two possible values: if $x(0) = 0$ then $C = 0$ but if $x(1) = 0$ then $C = -1$, a contradiction.

If this is not a valid solution, then the system:

$$\begin{cases} x''(t) = -x \\ x(0) = 0 \\ x(2\pi) = 1 \end{cases}$$

which has a general solution of the form $x(t) = C_1 \cos(x) + C_2 \sin(x)$ and again the constants cannot be properly determined: if $x(0) = 0$ then $C_1 = 0$ but if $x(2\pi) = 1$ then $C_1 = 1$, a contradiction.

Exercise 8.1

We classify the partial differential equations. For the linear equations we have:

Equation	Classification	Order
1	Linear	2
2	Linear	2
3	Linear	1
4	Linear	1
5	Linear	2
6	Linear	2
7	Linear	2
8	Linear	2
9	Linear	2
10	Linear	2
11	Linear	2
12	Linear	2
13	Linear	3
14	Linear	4

For the non linear equations we have:

Equation	Classification	Order
1	Fully Nonlinear	1
2	Semilinear	2
3	Quasilinear	2
4	Quasilinear	2
5	Fully Nonlinear	2
6	Fully Nonlinear	1
7	Quasilinear	1
8	Quasilinear	1
9	Semilinear	2
10	Quasilinear	2
11	Semilinear	2
12	Semilinear	3
13	Semilinear	2

Exercise 8.3

We prove the multinomial theorem by induction on the number of variables using the binomial theorem. Since this notation is horrid, we transform it to something understandable: given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$, having a sum over $|\alpha| = k$ is equivalent as having a sum over $\alpha_1 + \cdots + \alpha_n = k$, having x^α is having $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and having $\binom{|\alpha|}{\alpha}$ with $|\alpha| = k$ is having $k!/\alpha_1! \cdots \alpha_n!$, which is exactly $\binom{k}{\alpha_1,\ldots,\alpha_n}$. The dictionary above tells us that $\sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha = \sum_{\alpha_1+\cdots+\alpha_n=k} \binom{k}{\alpha_1,\ldots,\alpha_n} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We thus can just prove the usual binomial theorem (which is the right hand side), and the above will immediately give us the result we wanted.

When $n = 1$, we have:

$$x_1^k = \frac{k!}{k!0!} x_1^k = \sum_{\alpha_1 = k} \binom{\alpha_1}{\alpha_1} x_1^{\alpha_1}$$

as desired. Assume it is true for n, we check $n+1$:

$$\begin{aligned}
& (x_1 + \cdots + x_{n+1})^k = (x_1 + \cdots + (x_n + x_{n+1}))^k \\
= & \sum_{\alpha_1+\cdots+\alpha_{n-1}+\beta=k} \binom{k}{\alpha_1, \ldots, \alpha_{n-1}, \beta} x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} (x_n + x_{n+1})^\beta \\
= & \sum_{\alpha_1+\cdots+\alpha_{n-1}+\beta=k} \binom{k}{\alpha_1, \ldots, \alpha_{n-1}, \beta} x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \sum_{\alpha_n+\alpha_{n+1}=\beta} \binom{\beta}{\alpha_n, \alpha_{n+1}} x_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}} \\
= & \sum_{\alpha_1+\cdots+\alpha_{n-1}+\alpha_n+\alpha_{n+1}=k} \binom{k}{\alpha_1, \ldots, \alpha_{n-1}, \beta} \binom{\beta}{\alpha_n, \alpha_{n+1}} x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} x_{n+1}^{\alpha_{n+1}} \\
= & \sum_{\alpha_1+\cdots+\alpha_{n+1}=k} \binom{k}{\alpha_1, \ldots, \alpha_{n+1}} x_1^{\alpha_1} \cdots x_{n+1}^{\alpha_{n+1}}
\end{aligned}$$

which is exactly what we wanted. Thus, by the dictionary above, we just proved:

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$$

as desired.

Exercise 8.4

We prove Leibniz's formula. For this, we first note that given any two smooth functions $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$, restricting us to a single variable and using the rule of differentiation with respect of a multiplication, we obtain that for every $i = 1, \ldots, n$:

$$\partial_{x_i}^{\alpha_i}(fg) = \sum_{\gamma+\eta=\alpha_i} \binom{\alpha_i}{\gamma, \eta} \partial_{x_i}^{\gamma} f \partial_{x_i}^{\eta} g = \sum_{\gamma\leq\alpha_i} \binom{\alpha_i}{\gamma} \partial_{x_i}^{\gamma} f \partial_{x_i}^{\alpha_i-\gamma} g.$$

which is a result from Calculus with the notation adapted to our needs. We now abuse this for $u, v : \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth functions:

$$D^\alpha(uv) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}(uv) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-1}}^{\alpha_{n-1}} \left(\sum_{\beta_n\leq\alpha_n} \binom{\alpha_n}{\beta_n} \partial_{x_n}^{\beta_n} u \partial_{x_n}^{\alpha_n-\beta_n} v \right)$$

$$= \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{n-2}}^{\alpha_{n-2}} \left(\sum_{\beta_n\leq\alpha_n} \binom{\alpha_n}{\beta_n} \sum_{\beta_{n-1}\leq\alpha_{n-1}} \binom{\alpha_{n-1}}{\beta_{n-1}} \partial_{x_{n-1}}^{\beta_{n-1}} \partial_{x_n}^{\beta_n} u \partial_{x_{n-1}}^{\alpha_{n-1}-\beta_{n-1}} \partial_{x_n}^{\alpha_n-\beta_n} v \right)$$

where we have used $\partial_{x_n}^{\beta_n} f$ and $\partial_{x_n}^{\alpha_n-\beta_n} g$ as the new two functions. This process can be repeated until we obtain:

$$\begin{aligned}
D^\alpha(uv) &= \sum_{\beta_n\leq\alpha_n} \binom{\alpha_n}{\beta_n} \cdots \sum_{\beta_1\leq\alpha_1} \binom{\alpha_1}{\beta_1} \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} u \partial_{x_1}^{\alpha_1-\beta_1} \cdots \partial_{x_n}^{\alpha_n-\beta_n} v \\
&= \sum_{\substack{\beta_i\leq\alpha_i \\ i=1,\ldots,n}} \frac{\alpha_1!}{\beta_1!(\alpha_1-\beta_1)!} \cdots \frac{\alpha_n!}{\beta_n!(\alpha_n-\beta_n)!} D^\beta u D^{\alpha-\beta} v \\
&= \sum_{\beta\leq\alpha} \frac{\alpha_1! \cdots \alpha_n!}{\beta_1! \cdots \beta_n!(\alpha_1-\beta_1)! \cdots (\alpha_n-\beta_n)!} D^\beta u D^{\alpha-\beta} v \\
&= \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^\beta u D^{\alpha-\beta} v = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v
\end{aligned}$$

where we set $\beta = (\beta_1, \ldots, \beta_n)$. This is the result we wanted to prove.

Exercise 8.5

We prove Taylor's formula. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function, fix $x \in \mathbb{R}^n$ and consider $g : \mathbb{R} \longrightarrow \mathbb{R}$ given by $g(t) = f(tx)$. We can apply Taylor's one dimensional formula with residue around $t = 0$ and obtain:

$$g(t) = g(0) + g'(0) + \cdots + \frac{g^{(n)}(0)}{n!}t^n + \frac{g^{(n+1)}(\xi)}{(n+1)!}t^{n+1}.$$

By the chain rule, we have that:

$$\begin{aligned} g'(t) &= \frac{\partial f(tx)}{\partial x_1}\frac{\partial(tx_1)}{\partial t} + \cdots + \frac{\partial f(tx)}{\partial x_n}\frac{\partial(tx_n)}{\partial t} \\ &= \frac{\partial f(tx)}{\partial x_1}x_1 + \cdots + \frac{\partial f(tx)}{\partial x_n}x_n = (x_1, \cdots, x_n)\left(\frac{\partial f(tx)}{\partial x_1}, \cdots, \frac{\partial f(tx)}{\partial x_n}\right) \\ &= (x_1, \cdots, x_n)\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) f(tx) = xD_x f(tx) \end{aligned}$$

where we use a more convenient operator notation in this instance. Now, since x is fixed, it does not depend on time, and thus:

$$\begin{aligned} g^{(k)}(t) &= \frac{d^k f(tx)}{dt^k} = \frac{d^{k-1}(xD_x f(tx))}{dt^{k-1}} = (xD_x)\frac{d^{k-1} f(tx)}{dt^{k-1}} \\ &= \cdots = (xD_x)\cdots(xD_x)(xD_x f(tx)) = (xD_x)^k f(tx) \end{aligned}$$

since the derivatives commute. Since by the multinomial theorem we have:

$$\begin{aligned} (xD_x)^k &= \sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}(xD_x)^\alpha = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(x_1\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(x_n\frac{\partial}{\partial x_n}\right)^{\alpha_n} \\ &= \sum_{|\alpha|=k}\frac{k!}{\alpha!}x_1^{\alpha_1}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots x_n^{\alpha_n}\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = \sum_{|\alpha|=k}\frac{k!}{\alpha!}x_1^{\alpha_1}\cdots x_n^{\alpha_n}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \\ &= \sum_{|\alpha|=k}\frac{k!}{\alpha!}x^\alpha D_x^\alpha \end{aligned}$$

This means:

$$\begin{aligned} f(x) = g(1) = \sum_{k=0}^{n}\frac{g^{(k)}(0)}{k!}1^k + \frac{g^{(n+1)}(\xi)}{(n+1)!}1^{n+1} \\ = \sum_{k=0}^{n}\frac{1}{k!}\sum_{|\alpha|=k}\frac{k!}{\alpha!}x^\alpha D_x^\alpha f(0) + \frac{1}{(n+1)!}\sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}x^\alpha D_x^\alpha f(\xi x) \\ = \sum_{|\alpha|\le n}\frac{1}{\alpha!}x^\alpha D_x^\alpha f(0) + \sum_{|\alpha|=n+1}\frac{1}{\alpha!}x^\alpha D_x^\alpha f(\xi x) = \sum_{|\alpha|\le n}\frac{1}{\alpha!}x^\alpha D_x^\alpha f(0) + \mathcal{O}(|x|^{k+1}), \end{aligned}$$

because the last term behaves like x^{n+1} when x goes to 0 because the coefficient of x^α is $D_x^\alpha f(\xi x)/\alpha!$, which is differentiable in x thus continuous and with limit $D_x^\alpha f(0)/\alpha! \in \mathbb{R}$, not affecting the speed of the growth. This is the desired result.

References

[1] L. C. Evans, *Pertial Differential Equations - 2nd Edition*, American Mathematical Society, 2010.