Introduction to ODEs and PDEs - Homework 2

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Exercise 1 (1)[1]

We solve the initial value problem:

$$\begin{cases} u_t + bDu + cu = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = g \text{ in } \mathbb{R}^n \times \{t=0\} \end{cases}$$

for functions $u : \mathbb{R}^n \times (0,\infty) \longrightarrow \mathbb{R}$, $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ and constants $c \in \mathbb{R}$, $b \in \mathbb{R}^n$.

We notice that the multiplication bDu can be translated as the directional derivative of u in direction b, meaning that if we parametrize a general line with slope b, namely we fix $x \in \mathbb{R}^n$, $t \in (0,\infty)$ and consider the graph of $(x+bs, t+s)$ when $s \in \mathbb{R}$, then our initial value problem can be translated into a system of ordinary differential equations. For this, we set $z(s) = u(x+bs, t+s)$ and notice that:

$$\begin{aligned} \frac{\partial z(s)}{\partial s} &= Du(x+sb, t+s)b + u_t(x+sb, t+s) = -cu(x+sb, t+s) = -cz(s) \\ z(0) &= u(x,t) \end{aligned}$$

meaning that $z(s) = z(0)e^{-cs}$ so choosing $s = -t$ we obtain:

$$u(x,t) = u(x-bt, 0)e^{-ct} = g(x-bt)e^{-ct}$$

the formula we desired.

Exercise 2 (2)[1]

We prove that Laplace's equation is rotation invariant. Let A an orthogonal $n \times n$ matrix, $x \in \mathbb{R}^n$ with $\Delta u(x) = 0$ for certain $u : \mathbb{R}^n \longrightarrow \mathbb{R}$. Given $y = Ax$ we set $v(x) = u(y)$. We compute for each $1 \leq i \leq n$:

$$v_{x_i}(x) = u_{x_i}(y) = \sum_{j=1}^{n} \frac{\partial u(y)}{\partial y^j} \frac{\partial y^j}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial u(y)}{\partial y^j} a_{ji}$$

since $y^j = (Ax)^j = a_{j1}x_1 + \cdots + a_{jn}x_n$ for each $1 \leq j \leq n$. Moreover as above:

$$v_{x_i x_i} = \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial u(y)}{\partial y^j} a_{ji} \right) = \sum_{j=1}^{n} a_{ji} \sum_{k=1}^{n} \frac{\partial}{\partial y^k} \left(\frac{\partial u(y)}{\partial y^j} \right) \frac{\partial y^k}{\partial x_i} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 u(y)}{\partial y^k \partial y^j} a_{ji} a_{ki}$$

meaning that:

$$\begin{aligned} \Delta v(x) &= \sum_{i=1}^{n} v_{x_i x_i}(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 u(y)}{\partial y^k \partial y^j} a_{ji} a_{ki} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 u(y)}{\partial y^k \partial y^j} \sum_{i=1}^{n} a_{ji} a_{ki} \\ &= \sum_{j=1}^{n} \frac{\partial^2 u(y)}{\partial y^j \partial y^j} = \Delta u(y) = 0 \end{aligned}$$

because A orthogonal means $AA^T = 1$ hence $\sum_{i=1}^{n} a_{ji} a_{ki} = \delta_{jk}$ for all $1 \leq j, k \leq n$, and u is harmonic. This is what we wanted.

Exercise 3 (3)[1]

We modify the proof of the mean value formulas to obtain an particular expression for $u(0)$ when $n \geq 3$, provided $-\Delta u = f$ in $B(0,r)$ and $u = g$ in $\partial B(0,r)$.

In a similar way as indicated, we set:

$$\phi(r) = ⨍_{\partial B(x,r)} u(y) dS_y = ⨍_{\partial B(0,1)} u(x+rz) dS_z,$$

considering the change of variables $y = x + rz$. We have:

$$\begin{aligned}
\phi'(r) &= ⨍_{\partial B(0,1)} Du(x+rz) z dS_z = ⨍_{\partial B(x,r)} Du(y) \frac{y-x}{r} dS_y \\
&= ⨍_{\partial B(x,r)} Du(y)\nu dS_y = ⨍_{\partial B(x,r)} \frac{\partial u(y)}{\partial \nu} dS_y = \frac{r}{n} ⨍_{B(x,r)} \Delta u(y) dy
\end{aligned}$$

where $\nu = (y-x)/r$ is precisely the normal vector to the surface of the sphere, we used that $Du(y)\nu = \partial u(y)/\partial \nu$ is the definition of the directional derivative and the First Green Formula [1, Theorem 3 (p. 712)] guarantees the last equality. This is true for all $x \in \mathbb{R}^n$, we now choose $x = 0$ and note that:

$$\begin{aligned}
\phi(r) - \phi(\epsilon) &= \int_\epsilon^r \phi'(t) dt = \int_\epsilon^r \frac{t}{n} ⨍_{B(0,t)} \Delta u(y) dy dt = \int_\epsilon^r \frac{-1}{n\alpha(n) t^{n-1}} \int_{B(0,t)} f(y) dy dt \\
&= \frac{-1}{n\alpha(n)(2-n)} \left[\frac{1}{t^{n-2}} \int_{B(0,t)} f(y) dy \Bigg|_\epsilon^r - \int_\epsilon^r \frac{1}{t^{n-2}} \frac{\partial}{\partial t} \left(\int_{B(0,t)} f(y) dy \right) dt \right] \\
&= \frac{1}{n\alpha(n)(n-2)} \Bigg[\frac{1}{r^{n-2}} \int_{B(0,r)} f(y) dy - \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} f(y) dy \\
&- \int_\epsilon^r \frac{1}{t^{n-2}} \int_{\partial B(0,t)} f(y) dy dt \Bigg].
\end{aligned}$$

On the second line we have used the formula of integration by parts [1, Theorem 2 (p. 712)] with $u_{x_i} = 1/t^{n-1}$ so $u = 1/(2-n)s^{n-2}$ and $v = \int_{B(0,t)} f(y) dy$. On the third line we have used the differentiation formula for moving regions [1, Theorem 6 (p. 713)], resulting in:

$$\frac{\partial}{\partial t} \left(\int_{B(0,t)} f(y) dy \right) = \int_{\partial B(0,t)} f(y) \nu dS_y \int_{\partial B(0,t)} f(y) \nu \cdot \nu dy = \int_{\partial B(0,t)} f(y) dy$$

because the velocity of the moving boundary and the outward pointing unit normal coincide in these spheres. Now notice:

$$\begin{aligned}
\frac{1}{\epsilon^{n-2}}\int_{B(0,\epsilon)} f(y)dy &\leq \frac{1}{\epsilon^{n-2}}C\epsilon^n = C\epsilon^2 \overset{\epsilon\to 0}{\longrightarrow} 0\\
\int_\epsilon^r \frac{1}{t^{n-2}}\int_{\partial B(0,t)} f(y)dydt &\overset{\epsilon\to 0}{\longrightarrow} \int_0^r\int_{\partial B(0,t)} \frac{f(y)}{t^{n-2}}dydt = \int_{B(0,r)}\frac{f(x)}{|x|^{n-2}}dx\\
\phi(\epsilon) = ⨍_{\partial B(0,\epsilon)} u(y)dS_y &\overset{\epsilon\to 0}{\longrightarrow} u(0)\\
\phi(r) &= ⨍_{\partial B(0,r)} u(y)dS_y = ⨍_{\partial B(0,r)} g(y)dS_y
\end{aligned}$$

for C a constant whose existence is guaranteed by the continuity of f, and where we recall that an integral over a ball may be split into the consecutive integration of the surface of the sphere and the integration of the radius, and fixing a point $x \in B(0,r)$ the parameter t is precisely $d(0,x) = |x|$. The third line is justified by u being continuous, then we can obtain the value in a certain point by averaging the function in a neighborhood and making it small. Hence when ϵ goes to 0 we have that:

$$⨍_{\partial B(0,r)} g(y)dS_y - u(0) = \frac{1}{n\alpha(n)(n-2)}\left[\frac{1}{r^{n-2}}\int_{B(0,r)} f(y)dy - \int_{B(0,r)}\frac{f(x)}{|x|^{n-2}}dx\right]$$

so:

$$u(0) = ⨍_{\partial B(0,r)} g(y)dS_y + \frac{1}{n\alpha(n)(n-2)}\int_{B(0,r)}\left[\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right]f(x)dx$$

as desired.

Exercise 4 (5)[1]

1. Given $v : \overline{\Omega} \longrightarrow \mathbb{R}$ subharmonic, we proceed exactly as in the Exercise 3 above, defining:

$$\phi(r) = \fint_{\partial B(x,r)} v(y) dS_y,$$

and with the exact same reasoning we obtain:

$$\phi'(r) \;=\; \frac{r}{n} \fint_{B(x,r)} \Delta v(y) dy \geq 0$$

since everything is positive and v subharmonic guarantees $-\Delta v \leq 0$. This means that $\phi(r)$ is non-decreasing. Thus:

$$v(x) = \lim_{t \to 0} \fint_{\partial B(x,t)} v(y) dS_y = \lim_{t \to 0} \phi(t) \leq \phi(r) = \fint_{\partial B(x,r)} v(y) dS_y$$

where as before we average a continuous function in a neighborhood and make it small, and we use that since t is a radius (and ϕ is non-decreasing), we must have that the limit of a radius tending to zero must be equal or smaller than the value of the function in any other radius $r > 0$. Since this holds for all $B(x,r) \subset \Omega$, we obtain what we wanted.

2. Suppose that there is a point $x_0 \in \Omega$ with $v(x_0) = M = \max_{\overline{\Omega}} v$. Then for every $0 < r < d(x_0, \partial\Omega)$ we have by the above that:

$$v(x) \leq \fint_{\partial B(x,r)} v(y) dS_y \leq M,$$

so all these must be equalities and $v(y) = M$ for $y \in B(x_0, r)$. This means that the set $\{x \in \Omega : v(x) = M\}$ is open, and since it is a continuous preimage of a closed set, it is also closed. Hence if Ω is connected we must have $\Omega = \{x \in \Omega : v(x) = M\}$ and v is constant in Ω.

   Now, in the general case, we have two possibilities. If $\max_{\overline{\Omega}} v$ is attained in $\partial\Omega$, we are done. If $\max_{\overline{\Omega}} v \in \Omega$, by the above v is constant in the connected component where the maximum lies, meaning that in the boundary of that connected component the maximum value is also achieved. This yields $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$ as desired. Thus in both cases we have what we wanted.

3. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be smooth and convex, that is, $\psi''(x) \geq 0$ for every $x \in \mathbb{R}$. If u is

harmonic and $v(x) = \psi(u(x))$ we compute:

$$\begin{aligned}
-\Delta v(x) &= -\Delta\psi(u(x)) = -\sum_{i=1}^{n} \psi(u(x))_{x_i x_i} = -\sum_{i=1}^{n} (\psi'(u(x))u_{x_i}(x))_{x_i} \\
&= -\sum_{i=1}^{n} [\psi''(u(x))u_{x_i}(x)u_{x_i}(x) + \psi'(u(x))u_{x_i x_i}(x)] \\
&= -\sum_{i=1}^{n} \psi''(u(x))u_{x_i}(x)^2 - \psi'(u(x))\sum_{i=1}^{n} u_{x_i x_i}(x) \leq 0
\end{aligned}$$

since $\psi''(u(x))u_{x_i}(x)^2$ is multiplication of two positive terms for every $1 \leq i \leq n$ and $\sum_{i=1}^{n} u_{x_i x_i}(x) = \Delta u(x) = 0$. Thus v is subharmonic.

4. We consider now $v(x) = |Du(x)|^2$, we know that $Du(x) = (u_{x_1}(x), \ldots, u_{x_n}(x))$ so $|Du(x)|^2 = \sum_{i=1}^{n} u_{x_i}(x)^2$, meaning that:

$$\Delta |Du(x)|^2 = \sum_{i=1}^{n} \Delta u_{x_i}(x)^2$$

and for every $1 \leq i \leq n$ we have:

$$\begin{aligned}
\Delta u_{x_i}(x)^2 &= \sum_{j=1}^{n} (u_{x_i}(x)^2)_{x_j x_j} = \sum_{j=1}^{n} (2u_{x_i}(x)u_{x_i x_j}(x))_{x_j} = \sum_{j=1}^{n} [2u_{x_i x_j}(x)u_{x_i x_j}(x) \\
&+ 2u_{x_i}(x)u_{x_i x_j x_j}(x)] = 2\sum_{j=1}^{n} u_{x_i x_j}(x)^2 + 2u_{x_i}(x)\left(\sum_{j=1}^{n} u_{x_j x_j}(x)\right)_{x_i} \geq 0
\end{aligned}$$

since the first remaining term is positive as a sum of squares, and the second term is $2u_{x_i}(x)\left(\Delta u(x)\right)_{x_i} = 0$. This means that $\Delta v(x) \geq 0$ since it is simply a sum of positive terms, so $-\Delta v(x) \leq 0$ and v is subharmonic as desired.

Exercise 5 (10)[1]

1. Consider U^+ the open half ball, let $u \in \mathcal{C}^2(\overline{U^+})$ be harmonic in U^+ with $u(x) = 0$ on $\partial U^+ \cap \{x_n = 0\}$. We define:

$$v(x) = \begin{cases} u(x) \text{ if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) \text{ if } x_n < 0 \end{cases}$$

for $x \in U = B(0,1)$. We now proceed to prove that $v \in \mathcal{C}^\infty(U)$, which is more than we are asked, and that v is harmonic in U. For this, we note that v satisfies the mean value property in U^+ because it is harmonic in U^+. Fix now $x \in \mathbb{R}^n$ and consider $B(x,r) \subset U \setminus \overline{U^+}$, then:

$$\begin{aligned} ⨍_{\partial B(x,r)} v(y) dS_y &= ⨍_{\partial B(x,r)} -u(y_1, \ldots, y_{n-1}, -y_n) dS_y \\ &= -⨍_{\partial B((x_1,\ldots,x_{n-1},-x_n),r)} u(y_1, \ldots, y_{n-1}, y_n) dS_y \\ &= -u(x_1, \ldots, x_{n-1}, -x_n) = v(x) \end{aligned}$$

since now $\partial B((x_1, \ldots, x_{n-1}, -x_n), \delta) \subset U^+$ where the mean value property holds. Finally, if $x_n = 0$ we have that:

$$\begin{aligned} ⨍_{\partial B(x,r)} v(y) dS_y &= ⨍_{\partial B(x,r) \cap U^+} v(y) dS_y + ⨍_{\partial B(x,r) \cap (U \setminus \overline{U^+})} v(y) dS_y \\ &= ⨍_{\partial B(x,r) \cap U^+} u(y_1, \ldots, y_{n-1}, y_n) dS_y \\ &- ⨍_{\partial B(x,r) \cap (U \setminus \overline{U^+})} -u(y_1, \ldots, y_{n-1}, -y_n) dS_y \\ &= ⨍_{\partial B(x,r) \cap U^+} u(y_1, \ldots, y_{n-1}, y_n) dS_y \\ &- ⨍_{\partial B(x,r) \cap U^+} u(y_1, \ldots, y_{n-1}, y_n) dS_y = 0 = v(x) \end{aligned}$$

where we have used the symmetry of having $x_n = 0$, meaning that the integrals are exactly the same but with opposing sign. We also use that the difference of integrating over $\partial B((x_1, \ldots, x_{n-1}, x_n), r) \cap U^+$ and over $\partial B((x_1, \ldots, x_{n-1}, -x_n), r) \cap (U \setminus \overline{U^+})$ is $\{x \in U : x_n = 0\}$, a set of measure zero that does not affect the final value of the integral. This is the result for the average over the surface of balls, but the result for the integral over the balls follows in the same fashion that one proves the mean value formulas: we integrate over the radius, we can factor out the value $v(x)$ using the above and the only part missing is the volume ratio, which is exactly what we add when we integrate over the radius. Thus we have seen that for every ball $B(x,r) \subset U$ the function v satisfies the mean value property. Thus by [1, Theorem 6 (p. 28)] we obtain that $v \in \mathcal{C}^\infty(U)$, in particular $v \in \mathcal{C}^2(U)$. Moreover, by [1, Theorem 3 (p. 26)] we have that v is harmonic.

2. Assume now that $u \in \mathcal{C}^2(U^+) \cap \mathcal{C}(\overline{U^+})$, we want to show that v is harmonic within U. First we notice that with only these assumptions the statement is not true, since the function $u(x) = 1$ for every $x \in \overline{U^+}$, which is clearly smooth, yields a function v that is not even continuous, hence it cannot be harmonic. To prove what we are told we need that $u(x) = 0$ on $\partial U^+ \cap \{x_n = 0\}$ to guarantee that v is continuous and hence we are under the hypothesis of Poisson's formula for a ball [1, Theorem 15 (p. 41)]. Under these circumstances, we just apply said theorem to $U = B(0,1)$, obtaining:

$$w(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{v(y)}{|x-y|^n} dS_y$$

which by the Theorem is smooth in U, harmonic in U and for every $x_0 \in \partial B(0,1)$ we have:

$$v(x_0) = \lim_{\substack{x \to x_0 \\ x \in B(0,1)}} w(x) = w(x_0)$$

where the last equality is the definition that we can take of w in the boundary, which is guaranteed to be well defined since v is finite in the boundary. This means that we automatically obtain that $w(x) = v(x)$ for every $x \in \partial U$, meaning that $w(x) = v(x)$ in the whole $x \in U$ by the maximum principle. This means that v is harmonic, as desired.

Exercise 6 (11)[1]

We consider the Kelvin transform of a function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ as $\overline{u}(x) = u(\overline{x})|\overline{x}|^{n-2} = u(x/|x|)|x|^{2-n}$ for $x \neq 0$, where $\overline{x} = x/|x|^2$. We prove that if u is harmonic then $\overline{u}(x)$ is also harmonic.

We start by computing the basic derivatives. For every $1 \leq i \leq n$ we have:

$$\frac{\partial}{\partial}\left(\frac{1}{|x|^2}\right) = \frac{-2x_i}{|x^4|}$$

so that if we denote $\psi(x) = \overline{x} = (x_1, \ldots, x_n)/(x_1^2 + \cdots + x_n^2)$ so that we can easily work with their components, then:

$$\frac{\partial \psi^j(x)}{\partial x_i} = \frac{\delta_{ij}}{x_1^2 + \cdots + x_n^2} + x_j \frac{-2x_i}{|x|^4} = \frac{\delta_{ij}}{|x|^2} + x_j \frac{-2x_i}{|x|^4}.$$

This means that:

$$D_x\overline{x} = (\overline{x}_{x_1}, \ldots, \overline{x}_{x_n}) = \begin{bmatrix} \psi^1_{x_1} & \cdots & \psi^1_{x_n} \\ \vdots & & \vdots \\ \psi^n_{x_1} & \cdots & \psi^n x_n \end{bmatrix} = 1_{n\times n}\frac{1}{|x|^2} - \frac{2}{|x|^4}xx^T$$

since every $\overline{x}$ is a vector and thus its differential on every component is the one element-wise, yielding another vector matrix. Here we use that the multiplication $xx^T = (x_i x_j)_{ij}$ yields a matrix. It may seem unclear why we add this here, but this will be used in the following. We also note that $x^T x = x_1^2 + \cdots + x_n^2 = |x|^2$.

We continue computing derivatives; for every $1 \leq i, j \leq n$ we have:

$$\psi^j_{x_i x_i} = \frac{-2x_i}{|x|^4}\delta_{ij} - \frac{2x_i}{|x|^4}\delta_{ij} - 2x_j\frac{1}{|x|^4} - 2x_i x_j \frac{-4x_i}{|x|^6} = \frac{-4x_i\delta_{ij}}{|x|^4} - \frac{2x_j}{|x|^4} + \frac{8x_i^2 x_j}{|x|^6}$$

with:

$$\begin{aligned}
\Delta\psi^j &= \sum_{i=1}^n \psi^j_{x_i x_i} = \sum_{\substack{i=1 \\ i\neq j}}^n \left(\frac{-2x_j}{|x|^4} + \frac{8x_i^2 x_j}{|x|^6}\right) - \frac{2x_j}{|x|^4} - \frac{4x_j}{|x|^4} + \frac{8x_j^3}{|x|^6} \\
&= \frac{-(n-1+1+2)2x_j}{|x|^4} + \frac{8x_j}{|x|^6}\sum_{i=1}^n x_i^2 = (-2(n+2)+8)\frac{x_j}{|x|^4} = 2(2-n)\frac{x_j}{|x|^4}
\end{aligned}$$

and thus:

$$\Delta\psi = (\Delta\psi^i, \ldots, \Delta\psi^n) = 2(2-n)x\frac{1}{|x|^4}.$$

Equipped with these results, our goal is to compute $\Delta_x\overline{u}(x)$, and for this we use that for two scalar functions $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ we have $\Delta(fg) = f\Delta(g) + 2\Delta(f)\Delta(g) + g\Delta(f)$, which is a standard formula in vector calculus, yielding:

$$\begin{aligned}\Delta_x \overline{u}(x) &= \Delta\left(u\left(\frac{x}{|x|^2}\right)|x|^{2-n}\right) = \Delta_x(|x|^{2-n})u(\overline{x}) \\ &+ |x|^{2-n}\Delta_x\left(u\left(\frac{x}{|x|^2}\right)\right) + 2D_x u\left(\frac{x}{|x|^2}\right)D_x(|x|^{2-n}).\end{aligned}$$

Notice how the first term cancels out since at some point during the application of the chain rule since a term $\Delta_x(x) = 0$ will appear. For the middle term, we notice that if $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a vector field, then $\Delta_x(f \circ A) = \mathrm{Tr}(DA^T \cdot \Delta f\big|_{A(x)} \cdot DA) + Df\big|_{A(x)}^T \cdot \Delta A(x)$. In our particular application of this vector identity, we have $A = \overline{x}$ and $f = u$ harmonic, so the first summand is zero. Then:

$$\Delta_x \overline{u}(x) = |x|^{2-n} D_{\overline{x}} u \cdot \Delta_x \overline{x} + 2D_x u\left(\frac{x}{|x|^2}\right)D_x(|x|^{2-n}),$$

where we omit the transpose and we just recall the only proper way to multiply them (this allows us to loosely write the formulas and not have to worry about the non-commutativity of these multiplications). We will now work with the last term of the sum and check that it cancels the first. In particular, to compute $D_x(|x|^{2-n})$ we will need:

$$\frac{\partial |x|^{2-n}}{\partial x_i} = (2-n)|x|^{2-n-1}\frac{\partial |x|}{x_i} = (2-n)|x|^{2-n-1}\frac{x_i}{|x|} = (2-n)|x|^{-n}x_i$$

so:

$$D_x(|x|^{2-n}) = \left(\frac{\partial |x|^{2-n}}{\partial x_1}, \ldots, \frac{\partial |x|^{2-n}}{\partial x_n}\right) = (2-n)|x|^{-n}(x_1, \ldots, x_n).$$

This means that:

$$\begin{aligned}2D_x u\left(\frac{x}{|x|^2}\right)D_x(|x|^{2-n}) &= 2D_{\overline{x}} u \cdot D_x \overline{x} \cdot D_x(|x|^{2-n}) \\ &= 2D_{\overline{x}} u\left(1_{n\times n}\frac{1}{|x|^2} - \frac{2}{|x|^4}xx^T\right)(2-n)|x|^{-n}x \\ &= 2(2-n)|x|^{-n-2}D_{\overline{x}} u \cdot \left(1_{n\times n} - \frac{2xx^T}{|x|^2}\right)x \\ &= -2(2-n)|x|^{-2-n}D_{\overline{x}} u \cdot x = -|x|^{2-n}D_{\overline{x}} u \cdot \Delta_x \overline{x}.\end{aligned}$$

Hence:

$$\Delta_x \overline{u}(x) = -|x|^{2-n}D_{\overline{x}} u \cdot \Delta_x \overline{x} + |x|^{2-n}D_{\overline{x}} u \cdot \Delta_x \overline{x} = 0$$

and $\overline{u}$ is harmonic.

Exercise 7

1. We prove that any exponential function on $\mathbb{R}$ is an eigenfunction of all shifts. For this, let $f(x) = e^{ax}$ for $a, x \in \mathbb{R}$. We have that for all $t \in \mathbb{R}$:
$$f(x+t) = e^{a(x+t)} = e^{ax}e^{at} = \lambda_t f(x)$$
if we rename $\lambda_t = e^{at}$. This proves what we wanted.

2. We prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous is an eigenfunction of all shifts, then we must have $f(x) = Ce^{ax}$ for some constants $C, a \in \mathbb{R}$. For this, notice that if we have $f(x+t) = \lambda_t f(x)$, the assignment $g(t) = \lambda_t$ defines a function $g : \mathbb{R} \longrightarrow \mathbb{R}$. Moreover, if $f(x_0) = 0$ for certain $x_0 \in \mathbb{R}$, then $f(x) = f(x - x_0 + x_0) = \lambda_{x-x_0} f(x_0) = 0$ so the function is identically zero (that is, $C = 0$). Hence without loss of generality we may assume that the function f has constant sign and is never zero. Since $f(x+t) = g(t)f(x)$, this means that the function g must be positive to preserve the same sign on both sides of the equality and it is continuous as a fraction of continuous functions. We also have that for all $t, s \in \mathbb{R}$:
$$\begin{aligned} g(t+s)f(x) &= \lambda_{t+s} f(x) = f(x+t+s) = \lambda_s f(x+t) \\ &= \lambda_s \lambda_t f(x) = g(t)g(s)f(x) \end{aligned}$$
so $g(t+s) = g(t)g(s)$. We now prove that in fact $g(t) = g(1)^t$ for all $t \in \mathbb{R}$. We start with $n \in \mathbb{N}$, we have:
$$\begin{aligned} g(n) &= g(1 + \overset{n}{\cdots} + 1) = g(1) \overset{n}{\cdots} g(1) = g(1)^n, \\ g(-n) &= g(-1 - \overset{n}{\cdots} - 1) = g(-1) \overset{n}{\cdots} g(-1) = g(-1)^n = g(1)^{-n}, \end{aligned}$$
where we have used that $g(0) = \lambda_0 = 1$ since $f(x) = \lambda_0 f(x)$, implying that $g(1)g(-1) = g(1-1) = g(0) = 1$ and thus $g(-1) = g(1)^{-1}$. Now, for $n/m \in \mathbb{Q}$ we have:
$$g\left(\frac{1}{m}\right)^m = g\left(m\frac{1}{m}\right) = g(1) \quad \text{so} \quad g\left(\frac{1}{m}\right) = g(1)^{\frac{1}{m}},$$
$$g\left(\frac{n}{m}\right) = g\left(n\frac{1}{m}\right) = g\left(\frac{1}{m}\right)^n = \left(g(1)^{\frac{1}{m}}\right)^n = g(1)^{\frac{n}{m}},$$
where we heavily used that g is non-negative. Finally, if $t \in \mathbb{R}$, there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{Q}$ such that $\lim_{n\to\infty} t_n = t$, meaning that:
$$g(t) = g\left(\lim_{n\to\infty} t_n\right) = \lim_{n\to\infty} g(t_n) = \lim_{n\to\infty} g(1)^{t_n} = g(1)^{\lim_{n\to\infty} t_n} = g(1)^t$$
where we used that g and exponentiation are both continuous, so they commute with the limits. Now we have that for every $x, t \in \mathbb{R}$:
$$f(x+t) = g(t)f(x) = g(1)^t f(x) \text{ so } f(t) = f(0)g(1)^t = f(0)e^{t\log(g(1))}$$
by choosing $x = 0$. This clearly has the exponential form desired, with $C = f(0)$ and $a = \log(g(1))$.

References

[1] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.