# Introduction to ODEs and PDEs - Homework 3

Pablo Sánchez Ocal

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## Exercise 1 (12)[1]

Let $u : \mathbb{R}^n \times 0, \infty) \longrightarrow \mathbb{R}$ a smooth function with $u_t - \Delta u = 0$.

1. The function $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ satisfies $(u_\lambda)_t - \Delta(u_\lambda) = 0$ for every $\lambda \in \mathbb{R}$.

   This is seen by considering the change of variables $y = \lambda x$, $s = \lambda^2 t$ and computing:

$$\begin{aligned}
\frac{\partial u_\lambda(x,t)}{\partial t} &= \frac{\partial u(\lambda x, \lambda^2 t)}{\partial t} = \frac{\partial u(y,s)}{\partial t} = \frac{\partial u(y,s)}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial u(y,s)}{\partial s}\lambda^2 \\
\frac{\partial u_\lambda(x,t)}{\partial x_i} &= \frac{\partial u(\lambda x, \lambda^2 t)}{\partial x_i} = \frac{\partial u(y,s)}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial u(y,s)}{\partial y_j}\frac{\partial y_j}{\partial x_i} = \frac{\partial u(y,s)}{\partial y_i}\lambda \\
\frac{\partial}{\partial x_i}\left(\frac{\partial u_\lambda(x,t)}{\partial x_i}\right) &= \frac{\partial}{\partial x_i}\left(\frac{\partial u(\lambda x, \lambda^2 t)}{\partial x_i}\right) = \lambda \sum_{j=1}^{n} \frac{\partial}{\partial y_j}\left(\frac{\partial u(y,s)}{\partial y_i}\right)\frac{\partial y_j}{\partial x_i} \\
&= \lambda \frac{\partial^2 u(y,s)}{\partial y_i \partial y_i}\lambda
\end{aligned}$$

   where we have used the standard change of variables formula, noticing that the variables are independent with respect to each other. Moreover $\partial u(y,s)/\partial y_i = \partial u(x,t)/\partial x_i$ and $\partial u(y,s)/\partial s = \partial u(x,t)/\partial t$ since this differentiation only indicates that we are differentiating with respect to the second variable and the exact name of it is irrelevant. Thus:

$$\begin{aligned}
(u_\lambda)_t - \Delta(u_\lambda) &= \frac{\partial u_\lambda(x,t)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u_\lambda(x,t)}{\partial x_i}\right) \\
&= \lambda^2 \frac{\partial u(y,s)}{\partial s} - \sum_{i=1}^{n} \lambda^2 \frac{\partial^2 u(y,s)}{\partial y_i \partial y_i} = \lambda^2 (u_t - \Delta u) = 0
\end{aligned}$$

   since $u_t - \Delta u = 0$ by hypothesis. This is the desired result.

2. Use the section above to show that $v(x,t) = x \cdot Du(x,y) + 2tu_t(x,t)$ satisfies $v_t - \Delta v = 0$.

   One can directly check that $v(x,t)$ satisfies $v_t - \Delta v = 0$. However, we are asked to use the section above. Notice that:

$$\begin{aligned}
w(x,t,\lambda) &= \frac{\partial u_\lambda(x,t)}{\partial \lambda} = \frac{\partial u(\lambda x, \lambda^2 t)}{\partial \lambda} = \frac{\partial u(y,s)}{\partial t} \\
&= \frac{\partial u(y,s)}{\partial s}\frac{\partial s}{\partial \lambda} + \sum_{i=1}^{n} \frac{\partial u(y,s)}{\partial y_i}\frac{\partial y_i}{\partial \lambda} = 2\lambda t \frac{\partial u(y,s)}{\partial s} + \sum_{i=1}^{n} x_i \frac{\partial u(y,s)}{\partial y_i} \\
&= 2\lambda t u_t(x,t) + x \cdot Du(x,t)
\end{aligned}$$

   using what we noticed above that differentiation with respect to the first or second variables is independent of the exact name of those variables. Now clearly $w(x,t,\lambda) : \mathbb{R}^n \times (0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is smooth as composition of smooth functions, since $u$ is smooth and so is multiplication by $\lambda$. Moreover $w(x,t,1) = v(x,t)$, which will come into play later. Now, since we know:

$$\frac{\partial u_\lambda(x,t)}{\partial t} - \Delta u_\lambda(x,t) = 0$$

we can apply $\partial/\partial\lambda$, and the smoothness dependence guarantees that we can permute the differential operators, meaning that:

$$\begin{aligned} 0 &= \frac{\partial}{\partial\lambda}\left(\frac{\partial u_\lambda(x,t)}{\partial t} - \Delta u_\lambda(x,t)\right) = \frac{\partial}{\partial\lambda}\left(\frac{\partial u_\lambda(x,t)}{\partial t}\right) - \frac{\partial}{\partial\lambda}\left(\Delta u_\lambda(x,t)\right) \\ &= \frac{\partial}{\partial t}\left(\frac{\partial u_\lambda(x,t)}{\partial\lambda}\right) - \Delta\left(\frac{\partial u_\lambda(x,t)}{\partial\lambda}\right) = w_t(x,t,\lambda) - \Delta w(x,t,\lambda) \end{aligned}$$

which holds for every $\lambda \in \mathbb{R}$. In particular for $\lambda = 1$ we obtain:

$$0 = w_t(x,t,1) - \Delta w(x,t,1) = v(x,t) - \Delta v(x,t)$$

since, as noted above, $w(x,t,1) = v(x,t)$, and this is what we wanted.

## Exercise 2 (13)[1]

We assume $n = 1$ and $u(x,t) = v(x/\sqrt{t})$. Even if most if not all of the following is true for $t \in \mathbb{R}$, since we are in the context of the heat equation, we will consider $t \in (0, \infty)$.

1. We show that $u_t = u_{xx}$ if and only if $v'' + (z/2)v' = 0$, and then find the general solution of the latter.

   The first part is straightforward, since naming $z = x/\sqrt{t}$ we have:

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{1}{\sqrt{t}} \\
\frac{\partial z}{\partial t} &= \frac{-x}{2t^{3/2}} \\
u_t &= \frac{\partial v(z)}{\partial t} = v'\frac{\partial z}{\partial t} = v'\frac{-x}{2t^{3/2}} = v'\frac{x}{\sqrt{t}}\frac{1}{2}\frac{-1}{t} = v'\frac{z}{2}\frac{-1}{t} \\
u_x &= \frac{\partial v(z)}{\partial x} = v'\frac{\partial x}{\partial t} = v'\frac{1}{\sqrt{t}} \\
u_{xx} &= \frac{1}{\sqrt{t}}\frac{\partial v'(z)}{\partial x} = \frac{1}{\sqrt{t}}v''\frac{\partial x}{\partial t} = \frac{1}{\sqrt{t}}v''\frac{1}{\sqrt{t}}
\end{aligned}$$

   and thus:

$$\begin{aligned}
u_t = u_{xx} &\iff v'\frac{z}{2}\frac{-1}{t} = v''\frac{1}{t} \iff \frac{1}{t}v'' + \frac{1}{t}\frac{z}{2}v' = 0 \iff \frac{1}{t}\left(v'' + \frac{z}{2}v'\right) = 0 \\
&\iff v'' + \frac{z}{2}v' = 0
\end{aligned}$$

   since $1/t \neq 0$ for every $t \in (0, \infty)$. Every implication is an if and only if, proving what we desired.

   To compute the solution of $v'' + (z/2)v' = 0$, we first notice that if $v(z)$ has the form that we are told, then:

$$v' = ce^{-z^2/4} \quad \text{so} \quad e^{z^2/4}v' = c$$

   which means that $e^{z^2/4}v'(z)$ is constant as a function, so $(e^{z^2/4}v')' = 0$. If this ordinary differential equation is equivalent to our original ordinary differential equation (that is, they have exactly the same solutions), this would reduce the degree of our original ordinary differential equation, making it easier to solve. Hence this motivates us to ask if this is true. Assume $v$ is a solution of $v'' + (z/2)v' = 0$, then:

$$(e^{z^2/4}v')' = \frac{2z}{4}e^{z^2/4}v' + e^{z^2/4}v'' = e^{z^2/4}\left(v'' + \frac{z}{2}v'\right) = 0$$

   since by hypothesis $v'' + (z/2)v' = 0$. Assume now $v$ satisfies $(e^{z^2/4}v')' = 0$, then:

$$0 = (e^{z^2/4}v')' = \frac{2z}{4}e^{z^2/4}v' + e^{z^2/4}v'' = e^{z^2/4}\left(v'' + \frac{z}{2}v'\right)$$

which implies $v'' + (z/2)v' = 0$ since $e^{z^2/4} > 0$ for every $z \in \mathbb{R}$. This shows that a function $v$ satisfies $v'' + (z/2)v' = 0$ if and only if it satisfies $(e^{z^2/4}v')' = 0$. To compute the solutions of the latter, we can use separation of variables twice:

$$(e^{z^2/4}v')' = 0 \Longrightarrow e^{z^2/4}v' = c$$

for some constant $c \in \mathbb{R}$, and now:

$$e^{z^2/4}v' = c \Longrightarrow v' = ce^{-z^2/4} \Longrightarrow v = c\int_0^z e^{-s^2/4}ds + d$$

for some constant $c \in \mathbb{R}$. This is exactly the form that we wanted. To justify that this gives us a general expression, we can argue in two ways: the first is that we simply applied separation of variables, which indeed gives the general solution of an ordinary differential equation, and the second is that our solution has two parameters which are independent, that is, two degrees of freedom. Since our original equation is a second order differential equation, to determine a particular solution we need to determine two conditions, hence the degrees of freedom coincide and our solution is indeed the general form.

2. We compute $u_x$ and select $c$ accordingly to obtain the fundamental solution for the heat equation in dimension 1.

   Simply computing:

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial v(z)}{\partial x} = \frac{\partial v(z)}{\partial z}\frac{\partial z}{\partial x} = ce^{-z^2/4}\frac{1}{\sqrt{t}} = \frac{c}{\sqrt{t}}e^{-x^2/4t}.$$

   Notice that since $v$ is smooth by integration of a smooth function, then $u$ is smooth, meaning that the differential operators commute and $u_x$ is also a solution of the heat equation: since $v$ solves $v'' + (z/2)v' = 0$ then $u_t - u_{xx} = 0$, so:

$$0 = \frac{\partial}{\partial x}(u_t - u_{xx}) = u_{tx} - u_{xxx} = \frac{\partial u_x}{\partial t} - \frac{\partial^2 u_x}{\partial x \partial x}$$

   and $u_x$ is a solution of the heat equation in dimension 1. This will come into play later. Going back to computing $c$, we notice that we want the fundamental solution to integrate to 1, thus imposing for every fixed $t \in (0, \infty)$:

$$1 = \int_{\mathbb{R}} u_x(s)ds = \int_{\mathbb{R}} \frac{c}{\sqrt{t}}e^{-x^2/4t}dx = c\int_{\mathbb{R}} e^{-z^2/4}dz = c\sqrt{4\pi}$$

   where we used the change of variables $z = x/\sqrt{t}$ (which does not change the integration limits) yielding $dz = dx/\sqrt{t}$ and used that $\int_{\mathbb{R}} e^{-ax^2} = \sqrt{\pi}/\sqrt{a}$, which is a standard fact. This means that $c = 1/\sqrt{4\pi}$, and then indeed:

$$\frac{\partial u(x,t)}{\partial x} = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$$

is the fundamental solution to the heat equation in dimension 1.

The reason why this procedure gives the fundamental solution is because the way we computed this fundamental solution is assuming a function $w$ solved the heat equation and was of the form $\frac{1}{t^\alpha}\omega\left(\frac{x}{t^\beta}\right)$ for some function $\omega$ to be found, but satisfying some very concrete properties. Now, $u_x = \frac{1}{t^{1/2}}v'\left(\frac{x}{t^{1/2}}\right)$ is the solution that we proposed, which has choices $\alpha = 1/2 = \beta$, precisely the choices made when computing the fundamental solution for $n = 1$, and $v' = ce^{-z^2/4}$, which is also the condition that we imposed over $\omega$ when computing the fundamental solution. Hence our method is simply doing exactly the same thing that we did when computing the fundamental solution, except that we already have the optimal conditions instead of having to find them. Since we are doing exactly the same thing, we obtain exactly the same solution, which is, by definition, the fundamental solution.

## Exercise 3 (14)[1]

Given $c \in \mathbb{R}$, we want to find an explicit formula for a solution of:

$$\begin{cases} u_t - \Delta u + cu = f & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

For this, suppose we found one, say $u : \mathbb{R}^n \times (0, \infty) \longrightarrow \mathbb{R}$. Define now $v(x,t) = e^{ct}u(x,t)$ for $c \in \mathbb{R}$ the same one given by the original equation. Notice that $v$ is a solution of:

$$\begin{cases} v_t - \Delta v = e^{ct} f & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ v = g & \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

since:

$$\begin{aligned} \frac{\partial v(x,t)}{\partial t} - \Delta v(x,t) &= e^{ct} c u(x,t) + e^{ct} u_t(x,t) - e^{ct} \Delta u = e^{ct}(u_t - \Delta u + cu) = e^{ct} f, \\ v(x,0) &= e^0 u(x,0) = u(x,0) = g. \end{aligned}$$

Now by [1, Solution of non-homogeneous problem with general initial data (p. 51)], we have that:

$$v(x,t) = \int_{\mathbb{R}^n} \phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \phi(x-y, t-s) e^{cs} f(y,s) dy ds$$

where $\phi$ is the fundamental solution of the heat equation. This then gives us an explicit formula for $u$, which is what we want.

This motivates going in the reverse direction. We know that

$$v(x,t) = \int_{\mathbb{R}^n} \phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \phi(x-y, t-s) e^{cs} f(y,s) dy ds$$

where $\phi$ is the fundamental solution of the heat equation, solves:

$$\begin{cases} v_t - \Delta v = e^{ct} f & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ v = g & \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

by [1, Solution of non-homogeneous problem with general initial data (p. 51)]. Hence define $u(x,t) = e^{-ct} v(x,t)$, we have that:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) &= -e^{-ct} c v(x,t) + e^{-ct} v_t(x,t) - e^{-ct} \Delta v \\ &= -e^{-ct} c v(x,t) + e^{-ct}(v_t(x,t) - \Delta v) = e^{-ct} e^{ct} f - cu(x,t) \\ &= f - cu(x,t) \\ u(x,0) &= e^0 v(x,0) = v(x,0) = g \end{aligned}$$

and thus $u(x,t)$ is a solution of:

$$\begin{cases} u_t - \Delta u + cu = f & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u = g & \text{in} \quad \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

with explicit formula:

$$u(x,t) = e^{-ct} \int_{\mathbb{R}^n} \phi(x-y,t) g(y) dy + e^{-ct} \int_0^t \int_{\mathbb{R}^n} \phi(x-y, t-s) e^{cs} f(y,s) dy ds$$

as desired.

## Exercise 4 (16)[1]

We prove that if $\Omega$ is bounded and $u \in \mathcal{C}_1^2(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ solves the heat equation, then $\max_{\overline{\Omega_T}}(u) = \max_{\Gamma_T}(u)$, by using the hint provided: we consider $u_\epsilon = u - \epsilon t$ for $\epsilon > 0$, and we prove the above for $u_\epsilon$.

First, consider $v \in \mathcal{C}_1^2(\Omega_T) \cap \mathcal{C}(\overline{\Omega_T})$ satisfying $\Delta v - v_t > 0$. Then since $\overline{\Omega_T}$ is compact and $v$ continuous, there is a point $(x_0, t_0) \in \overline{\Omega_T}$ where $v$ attains its maximum. If $(x_0, t_0) \in \Omega_T$, then by hypothesis we can differentiate $v$, so the criterion that $(x_0, t_0)$ is a maximum if and only if the partial derivatives are zero applies, meaning that we have $v_t = 0$ and $v_{x_i} = 0$ for every $1 \leq i \leq n$, so we have $\Delta v = 0$. Then:

$$0 = \Delta v - v_t > 0,$$

a contradiction. Hence $(x_0, t_0) \in \overline{\Omega_T} \setminus \Omega_T = \Gamma_T$, meaning that $\max_{\overline{\Omega_T}}(v) = \max_{\Gamma_T}(v)$.

Now for $u_\epsilon$ we have:

$$\Delta u_\epsilon - \frac{\partial u_\epsilon}{\partial t} = \Delta u - (u_t - \epsilon) = \epsilon > 0$$

so the above reasoning applies, and we have $\max_{\overline{\Omega_T}}(u_\epsilon) = \max_{\Gamma_T}(u_\epsilon)$ for every $\epsilon > 0$. Now consider the sequence $\{u_{1/n}\}_{n \in \mathbb{N}}$, we have that:

$$|u_{1/n} - u| = \left| -\frac{t}{n} \right| \leq \frac{T}{n}$$

so given any $\delta > 0$, by choosing $N = \lceil (T/\delta) + 1 \rceil$ we have that for every $(x, t) \in \overline{\Omega_T}$ and all $n \geq N$ then $|u_{1/n} - u| \leq T/n \leq T/N < \delta$. This means that $\{u_{1/n}\}_{n \in \mathbb{N}}$ converges uniformly to $u$ in $\overline{\Omega_T}$, and in particular we can permute the limit and the supremum (which in this case, since $\overline{\Omega_T}$ is compact, is a maximum), obtaining:

$$\begin{aligned} \max_{\overline{\Omega_T}}(u) &= \max_{\overline{\Omega_T}} \lim_{n \to \infty} u_{1/n} = \lim_{n \to \infty} \max_{\overline{\Omega_T}} u_{1/n} \\ &= \lim_{n \to \infty} \max_{\Gamma_T} u_{1/n} = \max_{\Gamma_T} \lim_{n \to \infty} u_{1/n} = \max_{\Gamma_T}(u) \end{aligned}$$

the desired result.

## References

[1] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.