Introduction to ODEs and PDEs - Homework 4

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December 5th, 2017

Exercise 1

Let $g : \mathbb{R} \longrightarrow \mathbb{C}$ a compactly supported continuous function with Fourier transform $\tilde{g}(\xi)$. Find the Fourier transform of $e^{iax}g(x)$ where $a \in \mathbb{R}$ is a constant.

In this exercise and the following we will denote as $\mathcal{F}$ the Fourier transform operation, and given a function $g(x)$ we will denote $\mathcal{F}\{g(x)\}(\xi)$ its Fourier transform. This will prove more useful and illuminating than a tilde ˜ when dealing with Fourier transforms of translations and changes of variables of the original function. Hence in this notation, we are given $g(x)$, we know $\mathcal{F}\{g(x)\}(\xi)$ and we want to find $\mathcal{F}\{e^{iax}g(x)\}(\xi)$. By definition:

$$\mathcal{F}\{e^{iax}g(x)\}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} e^{iax} g(x) dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix(\xi-a)} g(x) dx = \mathcal{F}\{g(x)\}(\xi - a)$$

which by hypothesis is now in terms that we know.

Exercise 2

Let $g : \mathbb{R} \longrightarrow \mathbb{C}$ a compactly supported continuous function. Find the Fourier transform of $g(x+b)$ where $b \in \mathbb{R}$ is a constant. In the notation above, we compute:

$$\begin{aligned}
\mathcal{F}\{g(x+b)\}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} g(x+b)dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} e^{ib\xi} e^{-ib\xi} g(x+b)dx \\
&= \frac{e^{ib\xi}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(x+b)\xi} g(x+b)dx = \frac{e^{ib\xi}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\xi} g(y)dy \\
&= e^{ib\xi} \mathcal{F}\{g(y)\}(\xi) = e^{ib\xi} \mathcal{F}\{g(x)\}(\xi)
\end{aligned}$$

where we have used the change of variables $y = x + b$ so $dy = dx$, and that in the last equation the name of the variable is silent and does not carry any meaning.

Exercise 3

We derive by Fourier transform the d'Alembert formula for the solution of the initial value problem given by $u_{tt} - u_{xx} = 0$, $u|_{t=0} = g(x)$, $u_t|_{t=0} = 0$. We know that the d'Alembert formula for $u_{tt} - u_{xx} = 0$, $u|_{t=0} = g(x)$, $u_t|_{t=0} = h(x)$ is $u(x,t) = (1/2)[g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)dy]$. Because there are no shifts involved here, we will use the tilde notation.

Consider the given initial value problem, taking its Fourier transform on the variable x we obtain the initial value problem:

$$\begin{cases} \tilde{u}_{tt} + \xi^2 \tilde{u}_{xx} = \tilde{u}_{tt} - (i\xi)^2 \tilde{u}_{xx} = 0 \\ \tilde{u}|_{t=0} = \tilde{g} \\ \tilde{u}_t|_{t=0} = 0 \end{cases}$$

which is a system that we have seen multiple times and has for general solution $\tilde{u} = \tilde{g}\cos(t\xi) = (\tilde{g}/2)[e^{it\xi} + e^{-it\xi}]$.

Taking the inverse Fourier transform, we obtain:

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \frac{\tilde{g}(\xi)}{2} [e^{it\xi} + e^{-it\xi}] d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{2} [e^{-i(x-t)\xi} \tilde{g}(\xi) + e^{-i(x+t)\xi} \tilde{g}(\xi)] d\xi \\ &= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(x-t)\xi} \tilde{g}(\xi) d\xi + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i(x+t)\xi} \tilde{g}(\xi) d\xi \right] \\ &= \frac{1}{2} [\tilde{\tilde{g}}(x-t) + \tilde{\tilde{g}}(x+t)] = \frac{1}{2} [g(x+t) + g(x-t)] \end{aligned}$$

which is exactly the d'Alembert formula with h being the zero function.

Exercise 4

We derive Duhamel's principle for the intermediate value problem for the system of ordinary differential equations $u_{tt} = Au + f$, $u(0) = 0$, $u_t(0) = 0$, where $u : \mathbb{R} \longrightarrow \mathbb{C}^n$ and $A \in M_n(\mathbb{C})$ a fixed matrix.

What Duhamel's principle claims is that if $v(t; s)$ is a solution of the system of ordinary differential equations $v_t t = Av$, $v|_{t=s} = 0$, $v_t|_{t=s} = f(s)$ for a fixed $s \in \mathbb{R}$, then defining $u(t) = \int_0^t v(t; s)ds$, this is a solution of the original system. We prove that this is indeed a solution:

$$\begin{aligned}
u(0) &= \int_0^0 v(t; s)ds = 0 \\
u_t &= v(t; t) + \int_0^t v_t(t; s)ds = \int_0^t v_t(t; s)ds \\
u_t(0) &= \int_0^0 v_t(t; s)ds = 0 \\
u_{tt} &= v_t(t; t) + \int_0^t v_{tt}(t; s)ds = f + \int_0^t Av(t; s)ds = f + A\int_0^t v(t; s)ds = f + Au
\end{aligned}$$

where we simply applied the chain rule and the Fundamental Theorem of Calculus, and then used the hypothesis over v. Hence indeed u as defined is a solution of the original system, proving Duhamel's principle in this particular case.

Exercise 5

We are asked if there is a solution of $u_x + u_y = u$ whose graph contains the line $x = t$, $y = t$, $u = 1$, that we shall call L.

Assume we have a solution u, and that its graph indeed contains L parametrized by t as above. Looking at u along that line, we find that it is constant since $u(t, t) = 1$, and if we apply the differentiation rules:

$$0 = \frac{du}{dt}|_L = \frac{\partial x}{\partial t}|_L \frac{\partial u}{\partial x}|_L + \frac{\partial y}{\partial t}|_L \frac{\partial u}{\partial y}|_L = u_x|_L + u_y|_L = u|_L = 1$$

which is a contradiction. Hence, no such solution can exist.

References

[1] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.