Complex Variables I - Homework 1

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Exercise 1

We want to prove that every topological space X that is path connected is also connected.

We proceed by contradiction: suppose we can write $X = A \cup B$ with A, B open, non empty, $A \cap B = \emptyset$. Then take $x \in A$, $y \in B$ and since X is path connected, there exists a continuous $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x$, $\gamma(1) = y$; in particular $A \cap \gamma([0,1]) \neq \emptyset$ and $B \cap \gamma([0,1]) \neq \emptyset$. Since $[0,1]$ is connected and γ is continuous, $\gamma([0,1])$ is also connected, meaning that either $\gamma([0,1]) \subset A$ or $\gamma([0,1]) \subset B$ (entirely contained), rewritten; $B \cap \gamma([0,1]) = \emptyset$ or $A \cap \gamma([0,1]) = \emptyset$, a contradiction.

Exercise 2

We want to prove that a topological space X is compact if and only if every centered system of closed sets has non empty intersection.

$\Rightarrow$) We have X compact, take a centered system of closed sets $\{T_i\}_{i\in I}$. We proceed by contradiction: suppose $\bigcap_{i\in I} T_i = \emptyset$, then $X = (\bigcap_{i\in I} T_i)^C = \bigcup_{i\in I} T_i^C$. Since T_i^C is open for every $i \in I$, we obtain an open cover of X, thus by compactness, there is $N \subset I$ a finite subset with $X = \bigcup_{i\in N} T_i^C = (\bigcap_{i\in I} T_i)^C$, in particular $\emptyset = \bigcap_{i\in N} T_i$. This is a contradiction since $\{T_i\}_{i\in I}$ was a centered system of closed sets.

$\Leftarrow$) We have that in X every centered system of closed sets has non empty intersection. Take $\{U_i\}_{i\in I}$ an open cover: $X = \bigcup_{i\in I} U_i$, in particular it defines $\{U_i^C\}_{i\in I}$ a system of closed sets with $\emptyset = (\bigcup_{i\in I} U_i)^C = \bigcap_{i\in I} U_i^C$. We proceed by contradiction: suppose X is not compact, if we prove that $\{U_i^C\}_{i\in I}$ is a centered system, this immediately contradicts the original hypothesis. If X is not compact, then $X \neq \bigcup_{i\in N} U_i$ for every finite subset $N \subset I$, meaning that $\emptyset \neq (\bigcup_{i\in N} U_i)^C = \bigcap_{i\in N} U_i^C$, which shows that $\{U_i^C\}_{i\in I}$ has the finite intersections non empty and thus it is a centered system, as desired.

Exercise 3

We want to prove that whenever we have $K \subset X$ a compact subset of a Hausdorff space, then it is closed.

We prove that K^C is open. Let $x \in K^C$, for every $y \in K$ consider $U_y \ni x$ and $V_y \ni y$, whose existence is guaranteed by X being Hausdorff. Clearly $\{V_y\}_{y \in K}$ is an open cover of K, this being compact means that there exists $N \subset K$ such that $\{V_y\}_{y \in N}$ is an open cover. Now $x \in \bigcap_{x \in N} U_x$ which is open (since it is a finite intersection) and it does not intersect K since none of its components does, meaning that K^C is open.

Exercise 4

We want to prove that in a metric space (X, d) every convergent sequence is Cauchy.

A sequence $\{S_n\}_{n\in\mathbb{N}}$ converges to S when for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(S_n, S) \leq \varepsilon$ for $n > N$. A sequence $\{S_n\}_{n\in\mathbb{N}}$ is Cauchy when for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(S_n, S_m) > \varepsilon$ for $n, m > N$.

Consider $\{S_n\}_{n\in\mathbb{N}}$ a convergent sequence, then for $\varepsilon > 0$ take $N \in \mathbb{N}$ such that $d(S_n, S) \leq \varepsilon/2$. Then $d(S_n, S_m) \leq d(S_n, S) + d(S, S_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for $n, m > N$ and $\{S_n\}_{n\in\mathbb{N}}$ is Cauchy.

Exercise 5

We consider for (z_1, z_2, z_3, z_4) distinct complex numbers, the cross ratio:

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

Given any Möbius transformation, we want to prove that $[T(z_1), T(z_2), T(z_3), T(z_4)] = [z_1, z_2, z_3, z_4]$. Since any such T can be constructed via translations $b \in \mathbb{R}$, inversions and a scaling $c \in \mathbb{R}$, it is enough to prove that the cross ratio remains invariant by them:

1. $[T_b(z_1), T_b(z_2), T_b(z_3), T_b(z_4)] = \frac{(z_1+b-z_3-b)(z_2+b-z_4-b)}{(z_2+b-z_3-b)(z_1+b-z_4-b)} = [z_1, z_2, z_3, z_4]$,

2. $[I(z_1), I(z_2), I(z_3), I(z_4)] = \frac{\left(\frac{1}{z_1}-\frac{1}{z_3}\right)\left(\frac{1}{z_2}-\frac{1}{z_4}\right)}{\left(\frac{1}{z_2}-\frac{1}{z_3}\right)\left(\frac{1}{z_1}-\frac{1}{z_4}\right)} = \frac{\frac{z_3-z_1}{z_1z_3}\frac{z_4-z_2}{z_2z_4}}{\frac{z_3-z_2}{z_2z_3}\frac{z_4-z_1}{z1_z4}} = [z_1, z_2, z_3, z_4]$,

3. $[S_c(z_1), S_c(z_2), S_c(z_3), S_c(z_4)] = \frac{(cz_1-cz_3)(cz_2-cz_4)}{(cz_2-cz_3)(cz_1-cz_4)} = \frac{c(z_1-z_3)(z_2-z_4)}{c(z_2-z_3)(z_1-z_4)} = [z_1, z_2, z_3, z_4]$.

The first and third points being obvious, and the second not much more complicated: we just have to note that all the denominators in the small fractions cancel each other, and the remaining is a scaling by $c = -1$, which is invariant since we proved it in the third point (alternatively, one may prefer to simply factor out -1 and cancel it out).

Exercise 6

We want to prove that if (z_1, z_2, z_3, z_4) distinct belong to a circle, then $[z_1, z_2, z_3, z_4] \in \mathbb{R}$. For this, we will use T the Möbius transformation that takes:

$$\begin{array}{rcl} T(z_2) & \longrightarrow & 1 \\ T(z_3) & \longrightarrow & 0 \\ T(z_4) & \longrightarrow & \infty \end{array}$$

that we know exists since we can send any three points to any other three points. It can be readily checked that this transformation is:

$$T(z) = \frac{(z - z_3)(z_2 - z_4)}{(z_2 - z_3)(z - z_4)}.$$

We know that a Möbius transformation (thus T in particular) takes clines to clines, and since we take (z_2, z_3, z_4) to $(1, 0, \infty)$, the last three points being aligned along the real line, we must have that $T(z_1)$ belongs to the real line. We have the equality $[T(z_1), T(z_2), T(z_3), T(z_4)] = [z_1, z_2, z_3, z_4]$ proven above, the fact that the four original points are distinct guarantee that this is a finite value and the fact that after the transformation we belong in the real line guarantee that this value is a real one: a relation between real values is a real value.

Exercise 7

We have the Möbius transformation $c(z) = (z-i)/(z+i)$, we want to see that:

1. Takes i to 0: $z(i) = (i-i)/(i+i) = 0$.

2. The real axis to the unit circle, that is, for every $r \in \mathbb{R}$ we have $c(r) \in S^1$: $|c(r)| = |(r-i)/(r+i)| = |r-i|/|r+i| = |r-i|/|r-i| = 1$, enough for $c(r) \in S^1$.

3. The semicircle $z(\theta) = -b + \sqrt{1+b^2}\text{cis}(\theta)$, with $b \geq 0$ and $0 \leq \theta \leq \pi$, to a line segment in the unit circle passing through two antipodal points.

   Since we know by the point above that $c(z)$ maps the real axis to the unit circle, we look for the real values of $z(\theta)$. Those are $z(0) = -b+\sqrt{1+b^2}$ and $z(\pi) = -b-\sqrt{1+b^2}$. Those points are sent to $c(z(0)) = (-b+\sqrt{1+b^2}-i)/(-b+\sqrt{1+b^2}+i)$ and $c(z(\pi)) = (-b-\sqrt{1+b^2}-i)/(-b-\sqrt{1+b^2}+i)$, both clearly in S^1. We now check that they differ by a sign:

   $$\frac{c(z(0))}{c(z(\pi))} = \frac{-b+\sqrt{1+b^2}-i}{-b+\sqrt{1+b^2}+i} \cdot \frac{-b-\sqrt{1+b^2}+i}{-b-\sqrt{1+b^2}-i} = \frac{2i\sqrt{1+b^2}}{-2i\sqrt{1+b^2}} = -1.$$

   Thus $c(z(0)) = -c(z(\pi))$ and thus are antipodal in S^1.

   Finally, to verify that the image is indeed a straight line and not a circle, we will find the point that is sent to the 0, that is, we are looking for θ with $c(z(\theta)) = 0$. We impose:

   $$-b+\sqrt{1+b^2}(\cos(\theta)+i\sin(\theta)) - i = 0 \implies \begin{cases} -b+\sqrt{1+b^2}\cos(\theta) = 0 \\ \sqrt{1+b^2}\sin(\theta) - 1 = 0 \end{cases}$$

   and thus:

   $$\begin{cases} \cos(\theta) = \frac{b}{\sqrt{1+b^2}} \\ \sin(\theta) = \frac{1}{\sqrt{1+b^2}} \end{cases} \implies \theta_0 = \sin^{-1}\left(\frac{1}{\sqrt{1+b^2}}\right)$$

   which is a valid solution since $\cos(\sin^{-1}(\alpha)) = \sqrt{1-\alpha^2}$, thus $\cos(\theta_0) = b/\sqrt{1+b^2}$, and $1/\sqrt{1+b^2}$ is always positive and less than or equal to 1.

   Since we found a solution θ_0, we will have three aligned points $c(z(0))$, $c(z(\pi))$ and $c(z(\theta_0))$, thus the image will be a line segment, verifying the required conditions.

Exercise 8

We define:

$$\mathrm{SU}(1,1) = \left\{ g \in \mathrm{SL}_2(\mathbb{C}) : g^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

where g^* is the conjugate transpose of g. We prove that:

1. Given $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$, the equality:

$$\Delta = \left\{ z \in \mathbb{C} : \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} > 0 \right\}.$$

   This follows from the computation:

$$\begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{z} & 1 \end{pmatrix} \begin{pmatrix} -z \\ 1 \end{pmatrix} = 1 - |z|$$

   and thus:

$$z \in \Delta \iff 1 - |z| > 0 \iff z \in \left\{ z \in \mathbb{C} : \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} > 0 \right\}.$$

2. The action of $\mathrm{SU}(1,1)$ on $\mathbb{P}^1$ via Möbius transformations preserves the unit disk. This ask us to, given a point $z \in \Delta$ and an element $g \in \mathrm{SU}(1,1)$, verify that $gz \in \Delta$. We thus identify z as above and compute (using the properties above and the characterization of g):

$$\left(g \begin{pmatrix} z \\ 1 \end{pmatrix} \right)^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \left(g \begin{pmatrix} z \\ 1 \end{pmatrix} \right) = \begin{pmatrix} z \\ 1 \end{pmatrix}^* g^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} z \\ 1 \end{pmatrix} =$$
$$= \begin{pmatrix} z \\ 1 \end{pmatrix}^* \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} > 0 \Longrightarrow g \begin{pmatrix} z \\ 1 \end{pmatrix} \in \Delta.$$

   As mentioned, we have used indistinctly the characterization and the defining properties of $g \in \mathrm{SU}(1,1)$ and $z \in \Delta$.

3. Consider $a \in \Delta$ and $\phi_a(z) = (z-a)/(1-\overline{a}z)$ acting on $z \in \Delta$. To find its matrix form, we recall that:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix},$$

   the last equality being true in $\mathbb{P}^1$. Thus:

$$\phi_a = \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} 1 & -a \\ -\overline{a} & 1 \end{pmatrix}$$

   where the multiplying factor adjusts the matrix so that it has determinant one (we may observe that we can divide since $|a| < 1$ and that this multiplicative factor will not affect the final value of the operation since we are in $\mathbb{P}^1$ where having a non zero factor multiplying does not affect the result). Now:

$$\left(\frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix}1 & -a\\ -\overline{a} & 1\end{pmatrix}\right)^* \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\left(\frac{1}{\sqrt{1-|a|^2}}\begin{pmatrix}1 & -a\\ -\overline{a} & 1\end{pmatrix}\right) =$$

$$= \frac{1}{1-|a|^2}\begin{pmatrix}1 & -a\\ -\overline{a} & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -a\\ -\overline{a} & 1\end{pmatrix} =$$

$$= \frac{1}{1-|a|^2}\begin{pmatrix}-1 & -a\\ \overline{a} & 1\end{pmatrix}\begin{pmatrix}1 & -a\\ -\overline{a} & 1\end{pmatrix} = \frac{1}{1-|a|^2}\begin{pmatrix}-1+|a|^2 & 0\\ 0 & 1-|a|^2\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

and thus $\phi_a \in \mathrm{SU}(1,1)$.