# Complex Variables I - Homework 2

Pablo Sánchez Ocal

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## Exercise 1

Let $f$, $g$ be analytic functions on an open set $U \ni z_0$ with $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$. Then:

$$\begin{aligned}\lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \frac{z - z_0}{g(z) - g(z_0)} \\ &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} \frac{z - z_0}{g(z) - g(z_0)} = f'(z_0) \frac{1}{g'(z_0)}\end{aligned}$$

where we use the fact that when limits exist, the limit of a multiplication is the multiplication of limits. Since $g'(z_0) \neq 0$, this is well defined.

## Exercise 2

We derive the form of the Cauchy Riemann equations in polar coordinates. Note that since $x + iy = r(\cos(\theta) + i\sin(\theta))$ then $x = r\cos(theta)$, $y = r\sin(theta)$. Then when $f(z) = f(x, y) = u(x, y) + iv(x, y)$ we have:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{dx}{dr} + \frac{\partial u}{\partial y}\frac{dy}{dr} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$$
$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{dx}{d\theta} + \frac{\partial v}{\partial y}\frac{dy}{d\theta} = \frac{\partial v}{\partial x}(-\sin(\theta)) + \frac{\partial v}{\partial y}\cos(\theta) = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$$

and:

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{dx}{dr} + \frac{\partial v}{\partial y}\frac{dy}{dr} = \frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta)$$
$$\frac{-1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{dx}{d\theta} + \frac{\partial u}{\partial y}\frac{dy}{d\theta} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}(-\sin(\theta)) = \frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta)$$

where we have used the standard Cauchy Riemann equations to prove the equality. Thus:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}.$$

Note that $f(z) = z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$ and then:

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos(n\theta)$$
$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}r^n n\cos(n\theta) = nr^{n-1}\cos(n\theta)$$
$$\frac{\partial v}{\partial r} = nr^{n-1}\sin(n\theta)$$
$$\frac{-1}{r}\frac{\partial u}{\partial \theta} = \frac{-1}{r}r^n(-\sin(\theta)) = nr^{n-1}\sin(n\theta)$$

and $f(z)$ is holomorphic.

## Exercise 3

We construct a branch of $f(z) = \sqrt{z^2 - 1}$ on the complement of two half lines in $\mathbb{C}$. Let $s = z^2 - 1$, we know that $s^{1/2} = (e^{\log(s)})^{1/2} = e^{\log(s)/2}$ when $\log(s)$ is well defined. Now, $\log(t)$ for $t \in \mathbb{C}$ is well defined in $\mathbb{C} \setminus (-\infty, 0]$ since we need to remove 0 and a half line. Thus since $s = 0$ if and only if $z^2 = 1$ that is $z = \pm 1$, the change of variables $s = z^2 - 1$ translates 0 to $-1$ and adds $+1$ as a conflict. Thus we need to remove another half line to make $\log(s)$ well defined, say the line $[1, +\infty)$. This means that in $\mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$ (the complement of two half lines, as desired) we have that $\log(s)$ is well defined, and thus $f(z) = s^{1/2} = e^{\log(s)/2}$ is well defined.

## Exercise 4

Show that if $f(z) = f(2z)$ then $f(z)$ is constant: note that $f(z) = f(z/2^n)$ for any $n \in \mathbb{N}^+$. Since $f$ is continuous, for any $\varepsilon > 0$ given, there is a $\delta > 0$ such that if $|t-0| < \delta$ then $|f(t)-f(0)| < \varepsilon$. Choose $n$ with $|z/2^n| < \delta$, then $|f(z)-f(0)| = |f(z/2^n)-f(0)| < \varepsilon$ and then $f(z) = f(0)$ for every $z \in \mathbb{C}$.

If $f : \Delta \longrightarrow \mathbb{C}$ is continuous, the above remains true since the important point of argument is when considering points that are as close to $0 \in \Delta \subset \mathbb{C}$ as we desire.

## Exercise 5

Find the radius of convergence of:

1. $\sum_{n=0}^{\infty} a^{n^2} z^n$, $a > 0$: using the characterization given in Conway's book:

$$R = \lim_n \left| \frac{a_n}{a_{n+1}} \right| = \lim_n \left| \frac{a^{n^2}}{a^{(n+1)^2}} \right| = \lim_n \left| \frac{1}{a^{2n+1}} \right| = \begin{cases} 0 \text{ when } a > 1 \\ 1 \text{ when } a = 1 \\ +\infty \text{ when } a < 1 \end{cases}$$

2. $\sum_{n=0}^{\infty} z^{n!}$: which clearly diverges if $z = 1$, thus $R \leq 1$. If $|z| < 1$, then there is $r > 1$ with $|z| < 1/r < 1$ thus:

$$\sum_{n=0}^{\infty} |z|^{n!} < \sum_{n=0}^{\infty} \frac{1}{r^{n!}} < \sum_{n=0}^{\infty} \frac{1}{r^n}$$

which is the convergent geometric series. This means that $R = 1$.

3. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$: notice that:

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n} z^{n(n+1)} \right| = \sum_{n=0}^{\infty} \frac{|z^{n(n+1)}|}{n}$$

which diverges if $|z| = 1$, and thus we must have $R \leq 1$. When $|z| < 1$ using the root test:

$$\lim_n \left| \frac{z^{n(n+1)}}{n} \right|^{1/n} = \lim_n \frac{1}{n^{1/n}} \lim_n |z|^{n+1} = \lim_n |z|^n = 0$$

because $\lim_n n^{1/n} = 1$. Thus we have $R = 1$.

Consider now $z = 1$, we have $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ the harmonic series with a negative sign everywhere, thus convergent.

Consider now $z = 1$, we have $\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n^2} (-1)^n}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ since $n^2$ is odd or even when $n$ is odd or even respectively, thus $(-1)^{n^2} = (-1)^n$. This is the same harmonic series with a negative sign everywhere, thus convergent.

Consider now $z = i$, we have $S = \sum_{n=0}^{\infty} \frac{(-1)^n i^{n^2+n}}{n}$. Notice that $i^n$ takes the values $i$, $-1$, $-i$, $1$ and then repeats again, that is, it is 4 periodic. Moreover, $i^{n^2}$ is 2 periodic since $n^2 \equiv 1$ modulo 4 if $n$ is odd ($n \equiv 1$ implies $n^2 \equiv 1$ and $n \equiv 3$ implies $n^2 \equiv 9 \equiv 1$) and $n^2 \equiv 0$ modulo 4 if $n$ is even (since it contains at least two 2), meaning that $i^{n^2}$ takes the values $i$, $1$ and then repeats. Thus $i^{n^2+n}$ takes the values $-1$, $-1$, $1$, $1$ and then repeats, that is, it is 4 periodic. To prove it is convergent, we will compare it to $H$ the harmonic series:

$$H = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$
$$S = -1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \ldots$$

and then:

$$H + S = 2\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{6} + \frac{1}{7} - \dots\right)$$

since the signs are $+, -, +, -$ against $-, -, +, +$ meaning that we add the second and third terms of each 4, which correspond to an even and an odd number in the denominator respectively. Thus we have $H + S < 2H$ with $H$ being convergent, meaning that $S$ converges.

## Exercise 6

Let $\alpha : [0,1] \longrightarrow S^1$ be continuous, show that for any real number $\theta_0$ with $\alpha(0) = e^{i\theta_0}$ there exists a unique continuous map $\theta : [0,1] \longrightarrow \mathbb{R}$ with $\alpha(t) = e^{i\theta(t)}$, $\theta(0) = \theta_0$ for $t \in [0,1]$.

We have $\alpha(t) = e^{i\alpha} \in S^1$, let $\theta_0 = \varpi + 2\pi k$ for $\varphi \in [0, 2\pi)$ and $k \in \mathbb{Z}$ fixed (we can do this because we always have $e^{i\theta} = e^{i\varpi}$ for certain $\varpi \in [0, 2\pi)$ and their difference is a multiple of $2\pi$ by the properties of the exponential). We define:

$$\begin{array}{rccc} \theta & : & [0,1] & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \arg(\alpha(t)) + 2\pi k \end{array}$$

where $\arg(z) \in [0, 2\pi)$ is the argument of the complex number $z \in \mathbb{C}$.

This is well defined since if there are $t$, $t'$ with $\alpha(t) = \alpha(t')$ then $\arg(\alpha(t)) = \arg(\alpha(t'))$ since both are in $[0, 2\pi)$. Moreover $\alpha(t) = e^{i\arg(\alpha(t))} = e^{i\arg(\alpha(t))+i2\pi k} = e^{i\theta(t)}$, thus we only have to verify continuity. For this, we use the Lebesgue Covering Lemma, that we can apply since $S^1$ is compact. Given $\varepsilon < 0$, we want to find $\delta < 0$ such that when $|t - s| < \delta$ then $|\theta(t) - \theta(s)| < \varepsilon$. We have that $S^1 = \bigcup_{w \in S^1} B_\varepsilon(w)$, which is an open cover, and by the Lebesgue Covering Lemma there exists an $\delta > 0$ such that if $t \in S^1$ then $B_\delta(t) \subset B_\varepsilon(z)$ for some $z \in S^1$. Thus $s \in B_\delta(t)$ if and only if $|t - s| < \delta$ which implies $|\arg(\alpha(t)) - \arg(\alpha(t))| < \varepsilon$ by local continuity of $\arg(z)$ (in the whole $S^1$ we have that $\arg(z)$ is not continuous, but locally it is well defined, and such well definitions are coherent with the fact that adding any multiple of $2\pi$ yields the same angle in $S^1$) and $\alpha(t)$. We can use now this to see that $|\theta(t) - \theta(s)| = |\arg(\alpha(t)) - \arg(\alpha(t))| < \varepsilon$ and $\theta(t)$ is continuous.

Suppose we have $\beta : [0,1] \longrightarrow \mathbb{R}$ with the conditions above, then $e^{i\theta(t)} = \alpha(t) = e^{i\beta(t)}$ thus $\theta(t) = \beta(t) + 2\pi n$ for some $n \in \mathbb{Z}$, but since $\theta(0) = \beta(0)$, we must have $n = 0$ and $\beta(t) = \theta(t)$.

## Exercise 7

We are given three properties, we want to prove that there is a geodesic between any two points in $\Delta$. Consider $a, b \in \Delta$, then the map $\phi_a \in \mathrm{SU}(1,1)$ used in the Exercise 8 from the first problem set sends $a$ to 0 and $b$ to $\phi_a(b) \in \Delta$. Suppose $\phi_a(b) = re^{i\theta}$, then a multiplication by $e^{-i\theta}$ would send 0 to 0 and $\phi_a(b)$ to $r \in \mathbb{R}$. We note that we can write a multiplication $e^{-i\theta}z = \frac{e^{-i\theta}z+0}{0z+1} = \frac{e^{-i\theta/2}z+0}{0z+e^{i\theta/2}}$ meaning that it is represented by the matrix:

$$\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \text{ with } \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

that is, this multiplication is an element of $\mathrm{SU}(1,1)$. What we have now is 0 and $r$, both in the real line, but we know that $(-1,1)$ is a geodesic, and it goes through 0 and $r$, meaning that there is a geodesic between those two points. But since the operations we have used are elements of $\mathrm{SU}(1,1)$, in particular invertibles and preserving geodesics, the geodesic we found between 0 and $r$ is sent via $(e^{-i\theta} \cdot \phi_a)^{-1}$ to a geodesic from $a$ to $b$.

## Exercise 8

Show that relative to $\rho_\Delta$ the geodesics are always given by intersection with $\Delta$ of either a line or a circle which intersects the boundary of the disk at right angles.

Note that all the geodesics we found in the exercise above are via Möbius transformations of the geodesic $(-1,1)$, which is part of a line. Since Möbius transformations send clines to clines, any geodesic will be a cline (that is, a line or a cirlce) intersecting $\Delta$.

Moreover, note that since $\mathrm{SU}(1,1)$ sends geodesics to geodesics, and geodesics are locally length minimizing curves (and the distance between two points is the smallest length of any curve between them), we have that $\mathrm{SU}(1,1)$ must preserve distances. This means that $\mathrm{SU}(1,1)$ are isometries, in particular they preserve angles (an angle between vectors is defined relatively to the inner product and the length of vectors, and isometries preserve both since the inner product is given by the distance). Thus since $(-1,1)$ intersect the boundary of the disk at right angles, any other geodesic must also preserve such angle, thus they intersect the boundary of the disk at right angles.

## Exercise 9

Let $T$ be a non Euclidean triangle in $\Delta$ determined by the geodesic segments $A$, $B$, $C$ with intersections $A \cap C$, $B \cap C$, $A \cap B$ at angles $\alpha$, $\beta$, $\gamma$ respectively. Show that $\alpha + \beta + \gamma < \pi$.

Note that given the intersections, they cannot all three be in a straight line (unless all geodesic segments are in the same geodesic, but then $T$ is not a triangle). Thus, one of them must have real part smaller or bigger (or equal, in such a case we just choose one) than the rest. We suppose the case is the former (the latter results by an analogous reasoning) and that such a point is $A \cap C$ (by renaming the geodesics if necessary). We can assume that it has positive real part, since having negative real part will follow by an analogous reasoning.

The transformation $\phi_{A\cap C}$ sends $A \cap C$ to 0, $B \cap C$ to $\phi_{A\cap C}(B \cap C) = b$, $A \cap B$ to $\phi_{A\cap C}(A \cap B) = a$. For further commodity, by rotating the points if necessary, we may assume that $a$ lies on the real line and $b$ has positive complex part. This can be done since before rotating $b$ and $a$ must have different imaginary part (by the choice of $A \cap C$ having smallest positive real part), thus we rotate accordingly to what we want.

Note that after applying the steps above, the original geodesic segments $A$ and $C$ are transformed into geodesic segments $A'$, $C'$ obtained from $(-1, 1)$ by a simple rotation, since such are the form of the geodesics passing through 0 (that is, $A'$ and $C'$ are lines). Since this is done via elements of $\mathrm{SU}(1, 1)$, angles are preserved. We can now reason with 0, $b$, $a$ and the respective angles. Note that the geodesic segment $B'$ going from $b$ to $a$, that is, the one obtained from $B$, is a segment of a circle since it does not contain 0 and it must intersect with the boundary of the disk at right angles. Since we are in the side with positive real part and the circle obtained must intersect with the boundary of the disk at right angles, all points of such a segment of a circle must have real part smaller than the real part of the line $l$ going directly from $b$ to $a$ (except at the points $b$ and $a$, where $B'$ and $l$ coincide).

Now $A'$, $C'$, $l$ determine an Euclidean triangle and $A'$, $C'$, $B'$ determine a non Euclidean triangle with the same angles as the original $T$. But we have that the second triangle is strictly contained inside the first because $B'$ has real part smaller than $l$ and they coincide in the vertexes: thus the angles between $A'$ and $B'$, $C'$ and $B'$ are strictly smaller than the angles between $A'$ and $l$, $C'$ and $l$ respectively. This means that the sum of the angles of the second triangle must be strictly smaller. But since the first triangle is Euclidean, the sum of its angles is $\pi$, thus $\alpha + \beta + \gamma < \pi$, as desired.