Complex Variables I - Homework 3

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Problem 5 (p.74)

Give the power series expansion of $f(z) = \log(z)$ about $z = i$ and the radius of convergence.

Since $f(z)$ is analytic around $z = i$, we know that is has a power series expansion $f(z) = \sum_{n=0}^{\infty} f^{(n)}(i)(z-i)^n/n!$. Now:

$$f'(z) = \frac{1}{z}, \quad f''(z) = \frac{-1}{z^2}, \quad f'''(z) = \frac{2}{z^3}, \quad \dots \quad , \quad f^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{z^n},$$

and since for $n = 0$ we have $\log(i) = i\pi/2$, this means that:

$$f(z) = \frac{i\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{i^n} \frac{(z-i)^n}{n!} = \frac{i\pi}{2} - \sum_{n=1}^{\infty} \frac{i^n}{n!}(z-i)^n$$

since $1/i^n = (-1)^n i^n$.

Notice that since $f(0) = \log(0)$ is not well defined, we have a radius of convergence $R \leq 1$. However, since $f(z)$ is well defined in $B_1(i)$, by [1, Theorem 2.8 (p. 72)] we have that $R \geq 1$ and thus $R = 1$. This argument of computing the radius of convergence by simply finding an element where the power series does not converge but does for smaller radius will be used in the following problems without such a detailed explanation.

Problem 7 (p.74)

Evaluate the following integrals. In this exercise we will constantly use [1, Corollary 2.13 (p.73)], that is, $f^{(k)}(a) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-a)^{k+1}}$ where $\gamma(t) = a + re^{it}$ for $a \in \mathbb{C}$, $r \in \mathbb{R}^+$ and $0 \leq t < 2\pi$.

1. $\int_\gamma e^{iz}dz/z^2$, $\gamma(t) = e^{it}$, $0 \leq t < 2\pi$. Note that setting $f(z) = e^{iz}$ (analytic everywhere) we have $f'(z) = ie^{iz}$ thus:
$$\int_\gamma \frac{e^{iz}dz}{z^2} = 2\pi i f'(0) = -2\pi.$$

2. $\int_\gamma dz/(z-a)$, $\gamma(t) = a + re^{it}$, $0 \leq t < 2\pi$. Note that setting $f(z) = 1$ (analytic everywhere) we have:
$$\int_\gamma \frac{dz}{z-a} = 2\pi i f(a) = 2\pi i.$$

3. $\int_\gamma \sin(z)dz/z^3$, $\gamma(t) = e^{it}$, $0 \leq t < 2\pi$. Note that setting $f(z) = \sin(z)$ (analytic everywhere) we have $f''(z) = -\sin(z)$ thus:
$$\int_\gamma \frac{\sin(z)dz}{z^3} = \frac{2\pi i}{2}f''(0) = 0.$$

4. $\int_\gamma \log(z)dz/z^n$, $\gamma(t) = 1 + e^{it}/2$, $0 \leq t < 2\pi$, $n \geq 0$. Note that this integral is well defined on $\mathbb{C} \setminus (-\infty, 0)$, which includes $B_1/2(1)$. Hence $\log(z)dz/z^n$ is analytic where we want to integrate and γ is a closed path, by [1, Proposition 2.15 (p. 73)] we have:
$$\int_\gamma \frac{\log(z)dz}{z^3} = 0.$$

Problem 9 (p.75)

Evaluate the following integrals. As in the exercise above, we will constantly use [1, Corollary 2.13 (p.73)], that is, $f^{(k)}(a) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)dz}{(z-a)^{k+1}}$ where $\gamma(t) = a + re^{it}$ for $a \in \mathbb{C}$, $r \in \mathbb{R}^+$ and $0 \le t < 2\pi$.

1. $\int_\gamma (e^z - e^{-z})dz/z^n$, $\gamma(t) = e^{it}$, $0 \le t < 2\pi$. Note that setting $f(z) = e^z - e^{-z}$ (analytic everywhere) we have $f^{(n)}(z) = e^z + (-1)^{n-1}e^{-z}$ thus:

$$\int_\gamma \frac{(e^z - e^{-z})dz}{z^n} = \frac{2\pi i f^{(n-1)}(0)}{(n-1)!} = \frac{2\pi i(1+(-1)^n)}{(n-1)!} = \begin{cases} \frac{4\pi i}{(n-1)!} \text{ if } n \text{ even,} \\ 0 \text{ if } n \text{ odd.} \end{cases}$$

2. $\int_\gamma dz/(z-1/2)^n$, $\gamma(t) = 1/2 + e^{it}$, $0 \le t < 2\pi$. Note that setting $f(z) = 1$ (analytic everywhere) we have $f^{(n)}(z) = 0$ for $n > 1$ thus:

$$\int_\gamma \frac{dz}{(z-1/2)^n} = \frac{2\pi i f^{(n)}(1/2)}{n!} = \begin{cases} 2\pi i \text{ if } n = 1, \\ 0 \text{ if } n \neq 1. \end{cases}$$

3. $\int_\gamma dz/(z^2+1)$, $\gamma(t) = 2e^{it}$, $0 \le t < 2\pi$. For this, we will try to find A and B such that:

$$\frac{1}{z^2+1} = \frac{A}{z-i} + \frac{B}{z+i}\frac{Az + Ai + Bz - Bi}{z^2+1}$$

thus we have:

$$\begin{cases} A + B = 0 \\ Ai - Bi = 1 \end{cases} \Longrightarrow A = 1/2i,\ B = -1/2i.$$

Observe now that both $i, -i \in B_2(0)$ and setting $f(z) = 1$ (analytic everywhere) we have:

$$\int_\gamma \frac{dz}{z-i} = 2\pi i f(i) = 2\pi i = 2\pi i f(-i) = \int_\gamma \frac{dz}{z+i}.$$

This means that:

$$\int_\gamma \frac{dz}{z^2+1} = \int_\gamma \frac{1}{2i}\frac{dz}{z-i} - \int_\gamma \frac{1}{2i}\frac{dz}{z+i} = \frac{2\pi i}{2i} - \frac{2\pi i}{2i} = 0.$$

4. $\int_\gamma z^{1/m}dz/(z-1)^m$, $\gamma(t) = 1 + e^{it}/2$, $0 \le t < 2\pi$, $n \ge 0$. Set $f(z) = z^{1/m}$ (analytic everywhere), we have:

$$f'(z) = \frac{1}{m}z^{1/m-1}, \quad \ldots \quad , f^{(k)} = \left(\frac{1}{m}\right)\cdots\left(\frac{1}{m} - (k-1)\right)z^{1/m-k}.$$

Hence now:

$$\int_\gamma \frac{z^{1/m}dz}{(z-1)^m} = \frac{2\pi i f^{(m-1)}(1)}{(m-1)!} = \frac{2\pi i}{(m-1)!}\left(\frac{1}{m}\right)\cdots\left(\frac{1}{m} - (m-2)\right).$$

If we want to write the right hand side in a more compact way, we can say:

$$\int_\gamma \frac{z^{1/m}dz}{(z-1)^m} = \frac{2\pi i}{(m-1)!m^{m-1}} \prod_{j=0}^{m-2} (1-jm).$$

Problem 12 (p.75)

Show that $\sec(z) = 1 + \sum_{k=1}^{\infty} E_{2k} z^{2k}/(2k)!$, compute the radius of convergence, show that $E_{2n} - \binom{2n}{2n-2} E_{2n-2} + \cdots + \binom{2n}{2} E_2 + (-1)^n = 0$ and evaluate E_2, E_4, E_6, E_8.

We know that $\sec(z) = 1/\cos(z)$. This means that $\sec(z)$ is analytic in a neighborhood of 0 since $\sec(0) = 1$. Thus, we can expand as a power series $\sec(z) = \sum_{n=0}^{\infty} f^{(n)}(0) z^n / n!$. In particular by the above the term $n = 0$ is 1 thus:

$$\sec(z) = 1 + \sum_{n=1}^{\infty} \frac{f^{(n)}(0) z^n}{n!}.$$

Moreover, since $\cos(z)$ is an even function, $\sec(z)$ must be an even function, meaning that all the coefficients of the terms z^k with k odd must be zero. Hence:

$$\sec(z) = 1 + \sum_{n=1}^{\infty} \frac{f^{(2n)}(0) z^{2n}}{(2n)!},$$

the desired result by setting the constants $E_{2k} = f^{(2k)}(0)$. As mentioned and justified in an exercise above, the radius of convergence will be the closest distance to a non analytic point, that is, a point where $\cos(z) = 0$. The closest to 0 is $z = \pi/2$, thus the radius of convergence is $R = \pi/2$.

Now, since we have $1 = \cos(z)\sec(z)$ both analytic in $B_{\pi/2}(0)$, we can multiply the series and think of them as large polynomials:

$$1 = \left(\sum_{i=0}^{\infty} \frac{(-1)^i z^{2i}}{(2i)!} \right) \left(\sum_{j=0}^{\infty} \frac{E_{2j} z^{2j}}{(2j)!} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{(-1)^{k-j} E_{2j}}{(2i)!(2k-2j)!} \right) z^{2k}$$

where the way to see this is, first that the coefficients will stay powers of z^{2k} for k from 0 to ∞ since we are multiplying among them, second that by taking $k = i + j$ the coefficient of the term z^{2k}, we pick j from 0 to k from the second series and we then adjust for $i = k - j$ the term we will need from the first series. This same method of rearranging the multiplication will be used in the following exercise. For $k = 0$ we indeed have 1 (we are multiplying $1 \cdot 1$) and when $k > 0$ we need to have the coefficients equal to 0, thus we obtain:

$$0 = \sum_{j=0}^{k} \frac{(-1)^{k-j} E_{2j}}{(2i)!(2k-2j)!} = \sum_{j=0}^{k} (-1)^{k-j} E_{2j} \binom{2k}{2j}$$

by multiplying by $(2k)!$. This is precisely $E_{2n} - \binom{2n}{2n-2} E_{2n-2} + \cdots + \binom{2n}{2} E_2 + (-1)^n = 0$.

We notice that we already "found" $E_0 = 1$ for this to work, since the first term of the sec(z) expansion is 1. Now setting $k \in \{1,2,3,4\}$ we obtain:

$$\begin{aligned}
0 &= (-1) + E_2\binom{2}{2} \Longrightarrow E_2 = 1 \\
0 &= (+1) - 1\cdot\binom{4}{2} + E_4\binom{4}{4} \Longrightarrow E_4 = 5 \\
0 &= (-1) + 1\cdot\binom{6}{2} - 5\cdot\binom{6}{4} + E_6\binom{6}{6} \Longrightarrow E_6 = 61 \\
0 &= (+1) - 1\cdot\binom{8}{2} + 5\cdot\binom{8}{4} - 61\cdot\binom{8}{6} + E_8\binom{8}{8} \Longrightarrow E_8 = 1285.
\end{aligned}$$

Problem 13 (p.76)

Find the series expansion of $(e^z - 1)/z$ about 0 and determine its radius of convergence. Determine the radius of convergence of $f(z) = z/(e^z - 1) = \sum_{k=0}^{\infty} a_k z^k/k!$ and show that $a_0 + \binom{n+1}{1}a_1 + \cdots + \binom{n+1}{n}a_n = 0$. Show that $a_k = 0$ for $k > 1$ odd. Compute $B_{2n} = (-1)^{n-1}a_{2n}$ for $n \in \{1, 2, 3, 4, 5\}$.

Note that:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \Longrightarrow e^z - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!} \Longrightarrow \frac{e^z - 1}{z} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}$$

where the first two equations have an infinite radius of convergence. Now, since $(e^z - 1)/z \to 1$ when $z \to 0$ by L'Hospital rule, we have that we can extend $(e^z-1)/z$ by 1 for $z = 0$ and still obtain an analytic function, thus what we have done above is perfectly correct and the radius of convergence remains infinity.

For $f(z) = z/(e^z - 1)$, notice that the above remains true, thus for $z = 0$ we only have a removable singularity. The radius of convergence is then (as explained in the previous exercise) the distance from 0 to the nearest zeros of $e^z - 1$, which are $z = \pm 2\pi i$, hence $R = 2\pi$.

Now, by the same reasoning as before, consider:

$$1 = \frac{e^z - 1}{z}\frac{z}{e^z - 1} = \left(\sum_{i=0}^{\infty} \frac{z^i}{(i+1)!}\right)\left(\sum_{j=0}^{\infty} \frac{a_j z^j}{j!}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{a_j}{j!(k+1-j)!}\right) z^k$$

Again for $k = 0$ we indeed have 1 (we are multiplying $1 \cdot 1$) thus $a_0 = 1$ and when $k > 0$ we need to have the coefficients equal to 0, thus we obtain:

$$0 = \sum_{j=0}^{k} \frac{a_j}{j!(k+1-j)!} = \sum_{j=0}^{k} a_j \binom{k+1}{j}$$

by multiplying by $(k+1)!$. This is precisely $a_0 + \binom{k+1}{1}a_1 + \cdots + \binom{k+1}{k}a_k = 0$. In particular, we have:

$$0 = a_0\binom{2}{0} + a_1\binom{2}{1} = 1 + 2a_2 \Longrightarrow a_2 = \frac{-1}{2}.$$

Now, we are said that $f(z) + z/2$ is even, that is:

$$f(z) + \frac{z}{2} = \frac{z}{e^z - 1} + \frac{z}{2} = 1 + \frac{z}{2} + \frac{-z}{2} + \sum_{k=2}^{\infty} \frac{a_k z^k}{k!}$$

hence all the coefficients of the terms z^k with k odd must be zero, that is, $a_k = 0$ for $k > 1$ odd.

Set $B_{2n} = (-1)^{n-1} a_{2n}$, notice that:

$$\begin{aligned}
0 &= 1\binom{3}{0} - \frac{1}{2}\binom{3}{1} + a_2\binom{3}{2} \Longrightarrow a_2 = \frac{1}{6} \Longrightarrow B_2 = \frac{1}{6} \\
0 &= 1\binom{5}{0} - \frac{1}{2}\binom{5}{1} + \frac{1}{6}\binom{5}{2} + a_4\binom{5}{4} \Longrightarrow a_4 = \frac{-1}{30} \Longrightarrow B_4 = \frac{1}{30} \\
0 &= 1\binom{7}{0} - \frac{1}{2}\binom{7}{1} + \frac{1}{6}\binom{7}{2} + \frac{-1}{30}\binom{7}{4} + a_6\binom{7}{6} \Longrightarrow a_6 = \frac{1}{42} \Longrightarrow B_6 = \frac{1}{42} \\
0 &= 1\binom{9}{0} - \frac{1}{2}\binom{9}{1} + \frac{1}{6}\binom{9}{2} + \frac{-1}{30}\binom{9}{4} + \frac{1}{42}\binom{9}{6} + a_8\binom{9}{8} \Longrightarrow a_8 = \frac{-1}{30} \\
&\Longrightarrow B_8 = \frac{1}{30} \\
0 &= 1\binom{11}{0} - \frac{1}{2}\binom{11}{1} + \frac{1}{6}\binom{11}{2} + \frac{-1}{30}\binom{11}{4} + \frac{1}{42}\binom{11}{6} + \frac{-1}{30}\binom{11}{8} + a_{10}\binom{11}{10} \\
&\Longrightarrow a_{10} = \frac{5}{66} \Longrightarrow B_{10} = \frac{5}{66}.
\end{aligned}$$

A surprise that I encountered while working on this exercise is that here Conway we does not define the Bernoulli numbers in the usual way. In every convention, the a_k we computed are the usual Bernoulli numbers, since they are the coefficients of the expansion of $z/(e^z - 1)$, but here we change the sign of some of them to accommodate Conway's notation.

Problem 1 (p.80)

Let f be an entire function with $|f(z)| \leq M|z|^n$ for $|z| > R$, $n \in \mathbb{N}^+$, $M \in \mathbb{R}$ (all three fixed). Show that f is a polynomial of degree n or less.

Since f is entire, we know by [1, Proposition 3.3 (p. 77)] that:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

with infinite radius of convergence. Thus, it is enough to prove that $|f^{(k)}(0)| = 0$ for $k \geq n+1$, since then:

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

is a polynomial of the desired degree. Notice that setting $\gamma(t) = re^{it}$ for $r > R$ and $0 \leq t < 2\pi$ we have (by [1, Corollary 2.13 (p.73)]) for $k \geq n+1$:

$$|f^{(k)}(0)| = \left| \frac{k!}{2\pi i} \int_\gamma \frac{f(z)dz}{z^{k+1}} \right| \leq \frac{k!}{2\pi} \int_\gamma \frac{|f(z)||dz|}{|z^{k+1}|} \leq \frac{k!}{2\pi} \int_\gamma \frac{M|z|^n|dz|}{|z^{k+1}|} = \frac{k! 2\pi r}{2\pi r^{k+1-n}} = \frac{k! M}{r^{k-n}}$$

which goes to 0 as we increase $r \to \infty$ since $k - n \geq 1$. This proves $|f^{(k)}(0)| = 0$ which yields the desired result.

Problem 3 (p.80)

Find all entire functions with $f(x) = e^x$ for $x \in \mathbb{R}$. Notice that if f is entire, then:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

with infinite radius of convergence. In particular, for $x \in \mathbb{R}$ we have:

$$\sum_{k=0}^{\infty} \frac{1}{k!} z^k = e^x = f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

and since the radius of convergence is infinite, we can compare term by term and conclude that $f^{(k)}(0) = 1$ for every $k \in \mathbb{N}^+$, hence:

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k = e^z$$

by definition.

Notice that this can also be proven with [1, Corollary 3.8 (p. 79)] since the set $\{z \in \mathbb{C} : f(z) = e^z\}$ contains $\mathbb{R}$ and in particular 0, which is a limit point.

Problem 9 (p.80)

Let $u : \mathbb{C} \longrightarrow \mathbb{C}$ be a harmonic function with $u(z) \geq 0$ for every $z \in \mathbb{C}$. Prove that it is constant.

Since u is harmonic, by [1, Theorem 2.30 (p. 43)] there exists $v : \mathbb{C} \longrightarrow \mathbb{C}$ a harmonic function such that $f(z) = u(z) + iv(z)$ is entire (i.e. analytic in $\mathbb{C}$). In particular, the function $g(z) = e^{-f(z)}$ is entire since the exponential is entire. But now:

$$|g(z)| = |e^{u(z)}||e^{iv(z)}| = |e^{u(z)}| \leq 1$$

since both u and v take real values and u is positive or zero. Hence $g(z)$ is bounded and by Liouville's Theorem and thus constant. But this means that $f(z)$ is constant, meaning that we must have $u(z)$ constant, as desired.

References

[1] J. B. Conway, *Functions of One Complex Variable I*, Springer-Verlag, 2000.