Complex Variables I - Homework 4 (Midterm Exam)

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Problem 1

Let f be an entire function with $|f(z)| < \log(|z|)$ for $|z| > R$. We show that f is constant by proving that $f' \equiv 0$.

Let $z \in \mathbb{C}$ with $|z| > R$ and $\gamma(t) = z + re^{it}$ for $0 \leq t \leq 2\pi$ (choose r big enough such that $B_R(0)$ is inside $B_r(z)$), we use Cauchy's Integral Formula for the first derivative:

$$|f'(z)| = \left| \frac{1}{2\pi} \int_\gamma \frac{f(w)dw}{(w-z)^2} \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|f(w)||dw|}{|w-z|^2} \leq \frac{1}{2\pi} \int_\gamma \frac{\log(|z|)|dw|}{r^2} \leq \frac{1}{2\pi} \frac{\log(r)2\pi r}{r^2}$$

thus we have that:

$$|f'(z)| \leq \frac{\log(r)}{r} \to 0 \text{ when } r \to \infty.$$

Hence for some fixed $M > R$ we have that $|f'(z)| \leq 1$ when $|z| > M$. Moreover, since $\overline{B_M(0)}$ is a compact subspace and f' is continuous, it is bounded by some value, $|f'(z)| \leq N$ fixed. Hence f' is entire bounded in $\mathbb{C}$, thus by Liouville's Theorem f' is constant, but since f' takes arbitrary small values in module, we must have $f'(z) = 0$ for every $z \in \mathbb{C}$. This means that f is constant.

Problem 2

Consider $T(z) = (2z+3)/(z+2)$.

1. We clearly have that T fixes $\pm\sqrt{3}$:

$$T(\pm\sqrt{3}) = \frac{\pm 2\sqrt{3}+3}{\pm\sqrt{3}+2}\frac{\mp\sqrt{3}+2}{\mp\sqrt{3}+2} = \frac{-6 \pm 4\sqrt{3} \mp 3\sqrt{3}+6}{\mp 3+4} = \pm\sqrt{3}.$$

2. Moreover, T preserves the circle passing through $\sqrt{3}$, $-\sqrt{3}$, i since we will compute the cross ratio $[-\sqrt{3}, i, T(i), \sqrt{3}]$ and obtain a real number. This means that all four $\sqrt{3}$, $-\sqrt{3}$, i, $T(i)$ lie on the same circle. Since T fixes $\pm\sqrt{3}$, this means that the triplets $(\sqrt{3}, -\sqrt{3}, i)$ and $(\sqrt{3}, -\sqrt{3}, T(i))$ define the same circle, that is, T preserves the circle we want.

   For the computation, we have:

$$T(i) = \frac{2i+3}{i+2}\frac{i-2}{i-2} = \frac{i+8}{5}$$

   thus:

$$[-\sqrt{3}, i, T(i), \sqrt{3}] = \frac{-\sqrt{3}-\frac{i+8}{5}}{-2\sqrt{3}}\frac{i-\sqrt{3}}{i-\frac{i+8}{5}} = \frac{(5\sqrt{3}+i+8)(i-\sqrt{3})}{2\sqrt{3}(4i-8)}$$

   but since we are only interested in proving that this is real, not in computing the operation, we will divide and multiply by $(4i+8)$, which will turn the denominator into a real number, and we proceed with the numerator:

$$(5\sqrt{3}+i+8)(i-\sqrt{3})(4i+8) = (i(4\sqrt{3}+8) - 2(4\sqrt{3}+8))(4i+8)$$

   and now we only have to verify that the complex part of this number vanishes (we do not need to multiply the whole thing, just find the ones that contain i):

$$8i(4\sqrt{3}+8) - 8i(4\sqrt{3}+8) = 0$$

   as desired. Hence the cross ratio is real, and we obtain the desired result.

Problem 3

Let $p(z)$ be a polynomial of degree $n \in \mathbb{N}$, R big enough so that $B_R(0)$ contains every root of $p(z)$. Let $\gamma(t) = Re^{it}$ for $0 \leq t \leq 2\pi$, prove that:

$$\int_\gamma \frac{p'(z)}{p(z)} = 2n\pi i.$$

First, suppose that the zeros of $p(z)$ are $a_1, \ldots, a_m$ (all different) with respective multiplicities $k_1, \ldots, k_m$ (all non zero) with $m \leq n$ and obviously $n = k_1 + \cdots + k_m$. Now we know that $p(z) = (z - a_1)^{k_1} p_1(z)$ with the polynomial $p_1(a_1) \neq 0$, thus $p'(z) = k_1(z - a_1)^{k_1 - 1} p_1(z) + (z - a_1)^{k_1} p_1'(z)$. Moreover:

$$\frac{p'(z)}{p(z)} = \frac{k_1(z-a_1)^{k_1-1}p_1(z) + (z-a_1)^{k_1}p_1'(z)}{(z-a_1)^{k_1}p_1(z)} = \frac{k_1}{z-a_1} + \frac{p_1(z)}{p_1'(z)}. \tag{1}$$

This argument done m times implies that:

$$\frac{p'(z)}{p(z)} = \frac{k_1}{z-a_1} + \cdots + \frac{k_m}{z-a_m},$$

hence (we can separate the integral in its sums):

$$\int_\gamma \frac{p'(z)}{p(z)} = \int_\gamma \frac{k_1}{z-a_1} + \cdots + \int_\gamma \frac{k_m}{z-a_m} = 2\pi i k_1 + \cdots + 2\pi i k_m = 2n\pi i,$$

since we know that:

$$\int_\gamma \frac{1}{z-a} = 2\pi i \text{ when } a \text{ inside } \gamma.$$

Problem 4

Let $G \subset \mathbb{C}$ be open and connected, $a \in G$, $f : G \setminus \{a\} \longrightarrow \mathbb{C}$ holomorphic and f bounded on a neighborhood of a (say by M fixed). Let:

$$g(z) = \begin{cases} (z-a)^2 f(z) \text{ if } z \neq a \\ 0 \text{ if } z = a \end{cases}$$

we show:

1. Now $g'(a) = 0$:

$$|g'(a)| = |\lim_{h\to 0} \frac{g(a+h) - g(a)}{h}| = \lim_{h\to 0} \frac{|h^2||f(a+h)|}{|h|} \leq \lim_{h\to 0} |h|M = 0.$$

2. Knowing that g is analytic (by Goursat's Theorem), prove that f has a unique analytic extension to a holomorpic function on G. We define:

$$\tilde{f}(z) = \begin{cases} \frac{g(z)}{(z-a)^2} \text{ if } z \neq a \\ \frac{g^{(2)}(a)}{2} \text{ if } z = a \end{cases}$$

we clearly have that $\tilde{f}(z) = f(z)$ for $z \in \mathcal{G} \setminus \{a\}$, have to prove that $\tilde{f}$ is analytic. For this, we notice that since g is analytic and $g(a) = 0 = g'(a)$, we can write its series expansion as:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^n = (z-a)^2 \sum_{n=2}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^{n-2}$$

thus:

$$\tilde{f}(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(a)}{n!}(z-a)^{n-2},$$

which is analytic since it is continuously differentiable as a consequence of g being analytic. This proves that f has at least one analytic extension $\tilde{f}$.

Suppose we have another analytic function $\tilde{g}$ in G with $\tilde{g}(z) = f(z)$ in $G \setminus \{a\}$. This means that $\tilde{g}(z) = \tilde{f}(z)$ in $G \setminus \{a\}$, which is an open connected subspace of $\mathbb{C}$, hence $\tilde{g} \equiv \tilde{f}$ by [1, Corollary 3.9, p. 79].

Problem 5

Let f be entire with $f(\mathbb{C}) \subset \mathbb{C} \setminus [0,1]$. We show that f is constant. For this, we suppose f is entire non constant, we will achieve a contradiction. The idea is to prove that $|f(z)|$ is bounded for $|z| > R$ big enough, and thus bounded in $\mathbb{C}$ by being continuous in the compact $B_R(0)$, thus by Liouville's Theorem, we will obtain that f is constant.

Knowing that f is entire non constant, we now prove that $f(\mathbb{C})$ must be dense in $\mathbb{C}$. Suppose not, that is, there is a point $a \in \mathbb{C}$ such that for every $z \in \mathbb{C}$ we have that $|f(z) - a| > r$ fixed. Now we have:

$$\left| \frac{1}{f(z) - a} \right| = \frac{1}{|f(z) - a|} < \frac{1}{r}$$

for every $z \in \mathbb{C}$, thus since $1/(f(z) - a)$ is analytic since addition of analytic functions and inversion of an analytic function are analytic (where they are defined), and is bounded, by Liouville's Theorem $1/(f(z) - a)$ is constant hence $f(z)$ is constant. But this is a contradiction with f non constant. Hence the image is dense in $\mathbb{C}$.

Now, since the image of f is dense in $\mathbb{C}$, for every point $x \in [0,1]$ and every $\varepsilon > 0$ there is z_ε with $f(z_\varepsilon) \in B_\varepsilon(x)$ thus there is a convergent subsequence $\{f(z_i)\}_{i \in \mathbb{N}}$ with limit $x \in [0,1]$, that is:

$$\lim_{i \in \mathbb{N}} f(z_i) = x \iff f\left(\lim_{i \in \mathbb{N}} z_i\right) = x,$$

since f is continuous. Now $\lim_{i \in \mathbb{N}} z_i$ cannot converge since $x \notin f(\mathbb{C})$. Thus we obtain that when $f(z)$ tends to $[0,1]$ then $z \in \mathbb{C}$ tends to infinity. Since we can tend to $[0,1]$ from at least two different directions (and whose subsequences cannot intersect once we get close enough to 0 and 1 respectively since $\mathbb{C}$ is Hausdorff), namely tending to 0 from the left and to 1 from the right, we have control in the modulus $|f(z)| \leq 1$ when $|z| \to \infty$.[1]

This means that for R big enough, we have that if $|z| > R$ then $|f(z)| \leq 2$. Now $\overline{B_R(0)}$ is a compact bounded subspace, by f being continuous we have that f is bounded there, say $|f(z)| \leq M$ for $|z| \leq R$. Thus $|f(z)| \leq M + 2$ is a bound for $z \in \mathbb{C}$, hence by Liouville's Theorem we have that f is constant.

[1]I am not quite sure if the argument as presented here is correct. I feel that this approach can prove the result, but I am not sure if I untangled all the nuisances. Any help on this line of reasoning would be more than welcome.

Problem 5

Show that the function $f(t) = t/(e^t - 1)$ has a unique holomorphic extension to $B_1(0)$. Find the first four coefficients of the power series.

Notice that $f(t)$ is finite and well defined everywhere in $B_1(0)$ except maybe $t = 0$. We compute (using l'Hôpital's Rule):

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{t}{e^t - 1} = \lim_{t\to 0} \frac{1}{e^t} = 1,$$

thus f is bounded in $B_1(0)$. By Problem 4, f has a unique analytic extension to a holomorphic function on $B_1(0)$, say $\tilde{f}$. Letting:

$$\tilde{f}(t) = \sum_{n=0}^{\infty} a_n t^n,$$

we want to find a_0, a_1, a_2, a_3. We already computed $a_0 = \lim_{t\to 0} f(t) = 1$. Observe that:

$$e^t - 1 = \sum_{n=1}^{\infty} \frac{t^k}{k!}$$

thus the relation $\tilde{f}(t) = t/(e^t - 1)$ is transformed inside $B_1(0)$ to the power series relation (that can be treated as just giant polynomials since we are inside the radius of convergence):

$$(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)\left(t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots\right) = t,$$

hence we can compute the equality coefficient by coefficient after multiplying:

$$\begin{aligned} a_0 &= 1 \\ \frac{a_0}{2} + a_1 = 0 \Longrightarrow a_1 &= \frac{-1}{2} \\ \frac{a_0}{6} + \frac{a_1}{2} + a_2 = 0 \Longrightarrow a_2 &= \frac{1}{12} \\ \frac{a_0}{24} + \frac{a_1}{6} + \frac{a_2}{2} + a_3 = 0 \Longrightarrow a_3 &= 0 \end{aligned}$$

the coefficients we wanted.

References

[1] J. B. Conway, *Functions of One Complex Variable I*, Springer-Verlag, 2000.