Complex Variables I - Homework 5

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Problem 5 (p. 87)

Let γ be a close rectifiable curve in $\mathbb{C}$ and $a \notin \gamma$. Show that for $n \geq 2$ we have $\int_\gamma (z-a)^{-n} = 0$.

Notice that when $n \geq 2$ we can set $f(z) = (z-a)^{-n+1}/(-n+1)$ with:

$$f'(z) = \frac{(-n+1)(z-a^{-n+1-1})}{-n+1} = (z-a)^{-n+1},$$

thus $(z-a)^{-n}$ has a primitive, and letting γ have $\gamma(0) = \gamma(1)$ as initial and final point, by [1, Theorem 1.18 (p. 65)] we have that:

$$\int_\gamma \frac{1}{(z-a)^n} = f(\gamma(1)) - f(\gamma(0)) = 0,$$

as desired.

Problem 7 (p. 87)

Given $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$ compute $\int_\gamma z^n/(z-1)^n$ for $n \geq 1$.

Notice that since we integrate γ and $f(z) = z^n$ is entire with $f^{(n-1)}(z) = n!z$, by [1, Corollary 2.13 (p. 73)] we have that:

$$\int_\gamma \frac{z^n}{(z-1)^n} = \frac{2\pi i}{(n-1)!} f^{(n-1)}(1) = \frac{n!2\pi i}{(n-1)!} = n2\pi i.$$

Problem 8 (p. 87)

We have in a region G the family of analytic functions $\{f_n : G \longrightarrow \mathbb{C}\}_{n\geq 1}$ that converges uniformly to $f_G \longrightarrow \mathbb{C}$. We want to prove that f is analytic.

First, suppose $\mathbb{C} \setminus G$ is empty. Then for any triangle T in G we can directly apply [1, Cauchy's Integral Formula (p.84)] to the analytic function $g_n(z) = f_n(z)(z-a)$ for $n \geq 1$ and a fixed $a \in G \setminus T$ to obtain:

$$\int_T f(z) = \int_T \frac{f(z)(z-a)}{z-a} = \int_T \frac{g(z)}{z-a} = n(T,a)2\pi i g(a) = 0.$$

Secondly, pick any triangle T in G and suppose $\mathbb{C}\setminus G$ is nonempty. Then for $z \in \mathbb{C}\setminus G$ we have that z belongs to the unbounded component of $\mathbb{C} \setminus T$, thus $n(T,z) = 0$ by [1, Theorem 4.4 (p.82)] and hence we can again apply [1, Cauchy's Integral Formula (p.84)] to the analytic function $g_n(z) = f_n(z)(z-a)$ for $n \geq 1$ and a fixed $a \in G \setminus T$ to obtain:

$$\int_T f(z) = \int_T \frac{f(z)(z-a)}{z-a} = \int_T \frac{g(z)}{z-a} = n(T,a)2\pi i g(a) = 0.$$

Now, independently of the case above, we have by [1, Lemma 2.7 (p. 71)] that for ant triangle T in G:

$$\int_T f = \lim_n \int_T f_n = \lim_n 0 = 0,$$

thus by [1, Morera's Theorem (p. 86)] f is analytic in G.

Problem 4 (p. 95)

Let $G = \mathbb{C} \setminus \{0\}$, prove that every closed curve in G is homotopic to a closed curve whose trace is in $S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Given any closed curve $\gamma : [0,1] \longrightarrow G$, consider the curve $f : [0,1] \longrightarrow S^1$ given by $f(t) = \gamma(t)/|\gamma(t)|$ (obviously well defined since $|\gamma(t)/|\gamma(t)|| = |\gamma(t)|/|\gamma(t)| = 1$ and trivially closed when γ is closed), which is continuous by composition and division of continuous functions. We have that γ and f are homotopic by $H : [0,1] \times [0,1] \longrightarrow G$ given by $H(s,t) = s\gamma(t) + (1-s)f(t)$, which is obviously continuous by multiplication of continuous functions, and is and homotopy since $H(0,t) = f(t)$ and $H(1,t) = \gamma(t)$.

Thus we found the desired homotopy, proving the desired result.

Problem 8 (p. 96)

Let $G = \mathbb{C} \setminus \{a, b\}$ with $a \neq b$ points in $\mathbb{C}$ and γ as in the drawing.

1. Show that $n(\gamma, a) = 0 = n(\gamma, b)$. Notice that γ intersects itself three times, say on points L, C and R that staind for left, center and right, as pictured in the drawing.

   Consider L, following the curve's orientation to the right, we describe a (counterclockwise) loop that does not circle a and circles b once, and following the curve's orientation to the left, we describe a (clockwise) loop that does not circle a and circles b once. Hence γ does not circle a, thus $n(\gamma, a) = 0 + 0 = 0$, and circles b once counterclockwise and once clockwise, thus $n(\gamma, b) = +1 - 1 = 0$.

2. We will in fact prove that γ is not homotopic to zero. To do this, we will proceed in a similar way as the methods of "cutting and gluing" in Algebraic Topology, meaning that since we are dealing with a particular case and we are doing geometrical operations that are mathematically legal and in fact correspond to existing homotopies, the procedure is by all means a proof.

   Consider the point L and following the curve's orientation the segment that goes to C and then to R, starting on L, ending on L and only going through them once. Since this is a line segment, it is homotopic to zero, thus can be contracted to a single point. Thus our curve γ is homotopic in G to a curve that only intersects itself once (say on a point C) and has on the left side a counterclockwise loop (around the point a) inside a clockwise loop, and on the right side a clockwise loop (around the point b) inside a counterclockwise loop. Now, since the points a and b do not belong to G, we cannot continuously deform the inside loops to points, hence γ cannot be homotopic to zero.

Problem 11 (p. 96)

Evaluate $\int_\gamma (e^z - e^{-z})/z^4$ for the curves shown. First, notice that by [1, Definition 4.2 (p. 81)] and [1, Cauchy's Theorem (p. 90)] we have for any two homotopic rectifiable curves γ_1 and γ_2 (defined on any region G) that:

$$n(\gamma_1, a) = \frac{1}{2\pi i}\int_{\gamma_1}\frac{1}{z-a} = \frac{1}{2\pi i}\int_{\gamma_2}\frac{1}{z-a} = n(\gamma_2, a)$$

for every $a \in G$. Hence in our particular case, since we can set $G = \mathbb{C} \setminus \{0\}$ the domain of the function that we desire to integrate, we can fill any loops that do not circle 0 and our only concern to compute the integral is how many times each curve circles 0: we can make our curve homotopic to S^1 circling 0 as many times as needed, applying Problem 4 above if extreme rigorousness is called for. Such circling occur once (counterclockwise) for the first curve, twice (counterclockwise) for the second curve and also twice (counterclockwise) for the third curve (since two loops circle 0 in the same orientation and the third does not). Hence applying [1, Cauchy's Integral Formula (p.84)] with the justifications above, we consider $f(z) = e^z - e^{-z}$ with $f^{(3)}(z) = 2e^z$ and $f^{(3)}(0) = 2$, thus:

$$\int_\gamma \frac{e^z - e^{-z}}{z^4} = \frac{f^{(3)}(0)n(\gamma,0)2\pi i}{3!} = \frac{n(\gamma,0)2\pi i}{3},$$

now:

1. $n(\gamma_a, 0) = 1$ hence:
$$\int_\gamma \frac{e^z - e^{-z}}{z^4} = \frac{2\pi i}{3}.$$
2. $n(\gamma_b, 0) = 2$ hence:
$$\int_\gamma \frac{e^z - e^{-z}}{z^4} = \frac{4\pi i}{3}.$$
3. $n(\gamma_c, 0) = 2$ hence:
$$\int_\gamma \frac{e^z - e^{-z}}{z^4} = \frac{4\pi i}{3}.$$

References

[1] J. B. Conway, *Functions of One Complex Variable I*, Springer-Verlag, 2000.