Complex Variables I - Homework 6

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Problem 1 (p. 110)

Each of the following functions has an isolated singularity at $z = 0$. Determine if it is removable (and define $f(0)$ so that it becomes analytic), a pole (and find the singular part) or essential (and find $f(\{z : 0 < |z| < \delta\})$ for arbitrarily small $\delta \in \mathbb{R}$).

1. $f(z) = \sin(z)/z$, we compute:

$$\lim_{z\to 0} z\frac{\sin(z)}{z} = \lim_{z\to 0} \sin(z) = 0,$$

thus it is a removable singularity. Since moreover:

$$\lim_{z\to 0} \frac{\sin(z)}{z} = \lim_{z\to 0} \frac{\cos(z)}{1} = 1,$$

by l'Hôpital's rule, we can define $f(0) = 1$.

2. $f(z) = \cos(z)/z$, we compute:

$$\lim_{z\to 0} \left|\frac{\cos(z)}{z}\right| \leq \lim_{z\to 0} \frac{1}{|z|} = \infty,$$

thus it is a pole. Since moreover:

$$\frac{\cos(z)}{z} = \frac{1}{z}\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}\right) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k-1}}{(2k)!}$$

we found that the first term is the singular part and the second term is an analytic function.

3. $f(z) = (\cos(z) - 1)/z$, we compute:

$$\lim_{z\to 0} z\frac{\cos(z) - 1}{z} = \lim_{z\to 0} \cos(z) - 1 = 0,$$

thus it is a removable singularity. Since moreover:

$$\lim_{z\to 0} \frac{\cos(z) - 1}{z} = \lim_{z\to 0} \frac{\sin(z)}{1} = 0,$$

by l'Hôpital's rule, we can define $f(0) = 0$.

4. $f(z) = e^{1/z}$, since:

$$\lim_{x\to 0^+} xe^{1/x} = \infty, \quad \lim_{x\to 0^-} xe^{1/x} = 0,$$

and:

$$\lim_{x\to 0^+} e^{1/x} = \infty, \quad \lim_{x\to 0^-} xe^{1/x} = 0,$$

we have that it is an essential singularity. Now by [1, 4.2 Great Picard Theorem (p. 300)], since we cannot have $e^{1/z} = 0$ for any $z \in \mathbb{C}$, we have that in each neighborhood of $z = 0$ the function $f(z)$ assumes each complex number, except 0, an infinite number of times, thus $f(\{z : 0 < |z| < \delta\}) = \mathbb{C}\setminus\{0\}$ for arbitrarily small $\delta \in \mathbb{R}$. Another line of argument is saying that if we want to compute $f(1/z) = e^z$ for $|z| \geq 1/\delta$, no matter how small is δ, it always exists $n \in \mathbb{N}$ with $n > 1/\delta$, and now the set $\{z : ni \leq \mathrm{Im}(z) < 2\pi ni\}$ is mapped to $\mathbb{C} \setminus \{0\}$ via $f(1/z)$, and since we still cannot have $e^{1/z} = 0$ for any $z \in \mathbb{C}$, $f(\{z : 0 < |z| < \delta\}) = \mathbb{C} \setminus \{0\}$ for arbitrarily small $\delta \in \mathbb{R}$.

In this particular case we may go around the Great Picard Theorem, but in the following cases we may not be so lucky.

5. $f(z) = \log(z+1)/z^2$, we compute:

$$\lim_{z\to 0}\left|\frac{\log(z+1)}{z^2}\right| = \lim_{z\to 0}\frac{1}{|z+1|}\frac{1}{|2z|} = \infty,$$

thus it is a pole. Since moreover:

$$\frac{\log(z+1)}{z} = \frac{1}{z^2}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}z^k}{k!}\right) = \frac{1}{z} - \frac{1}{2} + \sum_{k=3}^{\infty}\frac{(-1)^{k+1}z^{k-2}}{k!}$$

we found that the first term $1/z$ is the singular part and the sum of second and third terms is an analytic function.

6. $f(z) = z\cos(1/z)$, since:

$$\lim_{x\to 0} x^2\cos(1/x) = 0, \quad \lim_{x\to 0}(ix)^2\cos(1/ix) = \lim_{x\to 0}(ix)^2\cosh(1/x) = \infty,$$

we obtain that $\lim_{z\to 0} z^2\cos(1/z)$ does not exist, and similarly:

$$\lim_{x\to 0}|x\cos(1/x)| = 0, \quad \lim_{x\to 0}|ix\cos(1/ix)| = \lim_{x\to 0}|x\cosh(1/x)| = \infty,$$

thus $\lim_{z\to 0}|z\cos(1/z)|$ does not exist and we have that it is an essential singularity. An alternative way of seeing this is that:

$$z\cos(1/z) = z\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!z^{2k}}\right) = z + \sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!z^{2k-1}}$$

with infinitely many negative powers of z, meaning that we cannot have a removable singularity nor a pole (hence it can only be an essential singularity). Now by [1, 4.2 Great Picard Theorem (p. 300)], we have that in each neighborhood of $z = 0$ the function $f(z)$ assumes each complex number, with one possible exception, an infinite number of times. However, notice that $f(z)$ is odd since $f(-z) = (-z)\cos(-1/z) = -z\cos(1/z) = -f(z)$. Thus if there is $a \in \mathbb{C}$ such that $f(z) = a$ has only a finite number of solutions (maybe none), the equation $f(z) = -a$ must have (by the above theorem) an infinite number of solutions, say $\{z_j\}_{j\in J}$. Then $f(-z_j) = -f(z_j) = a$, and we have $\{-z_j\}_{j\in J}$ an infinite number of solutions. Thus the only possible point where this may occur is $a = 0$. However, note that defining $x_n = 1/(n\pi - pi/2)$ for $n \in \mathbb{Z}$ we have $f(x_n) = x_n \cos(1/x_n) = 0$, and taking n as big as we need, we have infinite $x_n \in \{z : 0 < |z| < \delta\}$ for arbitrarily small $\delta \in \mathbb{R}$. This means that $f(\{z : 0 < |z| < \delta\}) = \mathbb{C}$ for arbitrarily small $\delta \in \mathbb{R}$.

7. $f(z) = (z^2 + 1)/z(z - 1)$, we compute:

$$\lim_{z\to 0} \left| \frac{z^2+1}{z(z-1)} \right| = \infty, \quad \lim_{z\to 1} \left| \frac{z^2+1}{z(z-1)} \right| = \infty,$$

thus there are poles at $z = 0$ and $z = 1$. We now decompose the function into fractions (we will spare the reader the whole computation, once done, it is easily verified that the following decomposition is correct):

$$\frac{z^2+1}{z(z-1)} = 1 + \frac{z+1}{z(z-1)} = 1 - \frac{1}{z} + \frac{2}{z-1},$$

hence $-1/z$ is the singular part at $z = 0$ and $2/z - 1$ is the singular part at $z = 1$.

8. $f(z) = 1/(1 - e^z)$, we compute:

$$\lim_{z\to 0} \left| \frac{1}{1-e^z} \right| = \infty,$$

thus it is a pole. Notice that if we define $g(z) = zf(z) = z/(1 - e^z)$, then:

$$\lim_{z\to 0} z\frac{z}{1-e^z} = \lim_{z\to 0} \frac{2z}{-e^z} = 0,$$

by l'Hôpital's rule, thus $g(z)$ has a removable singularity at $z = 0$ and since:

$$\lim_{z\to 0} \frac{z}{1-e^z} = \lim_{z\to 0} \frac{1}{-e^z} = -1,$$

by l'Hôpital's rule, we can define $g(0) = -1$. This means that near (but not in) $z = 0$ we have $f(z) = g(z)/z$ or by the above, the first term in the power series expansion near $z = 0$ of $g(z)$ is -1, thus $-1/z$ is the singular part of $f(z)$ at $z = 0$.

9. $f(z) = z\sin(1/z)$, since:

$$\lim_{x\to 0} x^2 \sin(1/x) = 0, \quad \lim_{x\to 0} |(ix)^2 \sin(1/ix)| = \lim_{x\to 0} |(ix)^2 i \sinh(1/x)| = \infty,$$

we obtain that $\lim_{z\to 0} z^2 \sin(1/z)$ does not exist, and similarly:

$$\lim_{x\to 0} |x\sin(1/x)| = 0, \quad \lim_{x\to 0} |ix \sin(1/ix)| = \lim_{x\to 0} |xi \sinh(1/x)| = \infty,$$

thus $\lim_{z\to 0}|z\sin(1/z)|$ does not exist and we have that it is an essential singularity. An alternative way of seeing this is that:

$$z\sin(1/z) = z\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!z^{2k+1}}\right) = 1+\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)!z^{2k}}$$

with infinitely many negative powers of z, meaning that we cannot have a removable singularity nor a pole (hence it can only be an essential singularity). Moreover, [2] proves that the equation $\sin(z) = az$ has infinitely many roots for every complex number $a \in \mathbb{C}$, and thus by [1, Theorem 3.7 (p. 78)] we need that those solutions cannot be contained in any circle, otherwise said, given any $\delta \in \mathbb{R}$ we need the existence of a root s with $|s| > 1/\delta$. Applying this to the equation $f(1/z) = a$, we obtain that there are solutions in every $\{z : 0 < |z| < \delta\}$ for arbitrarily small $\delta \in \mathbb{R}$ and every $a \in \mathbb{C}$, that is, $f(\{z : 0 < |z| < \delta\}) = \mathbb{C}$ for arbitrarily small $\delta \in \mathbb{R}$.

10. $f(z) = z^n\sin(1/z)$, in an analogous fashion as before:

$$\lim_{x\to 0} x^{n+1}\sin(1/x) = 0, \quad \lim_{x\to 0}|(ix)^{n+1}\sin(1/ix)| = \lim_{x\to 0}|(ix)^{n+1}i\sinh(1/x)| = \infty,$$

we obtain that $\lim_{z\to 0} z^n\sin(1/z)$ does not exist, and similarly:

$$\lim_{x\to 0}|x^n\sin(1/x)| = 0, \quad \lim_{x\to 0}|(ix)^n\sin(1/ix)| = \lim_{x\to 0}|x^n i\sinh(1/x)| = \infty,$$

thus $\lim_{z\to 0}|z^n\sin(1/z)|$ does not exist and we have that it is an essential singularity. An alternative way of seeing this is that:

$$z\sin(1/z) = z^n\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!z^{2k+1}}\right)$$

whose simplification depends on whether n is even or odd, but in both cases has infinitely many negative powers of z, meaning that we cannot have a removable singularity nor a pole (hence it can only be an essential singularity).

Suppose n is even. Now by [1, 4.2 Great Picard Theorem (p. 300)], we have that in each neighborhood of $z = 0$ the function $f(z)$ assumes each complex number, with one possible exception, an infinite number of times. However, notice that $f(z)$ is odd since n is even and thus $f(-z) = (-z)^n\sin(-1/z) = -z\sin(1/z) = -f(z)$. Thus if there is $a \in \mathbb{C}$ such that $f(z) = a$ has only a finite number of solutions (maybe none), the equation $f(z) = -a$ must have (by the above theorem) an infinite number of solutions, say $\{z_j\}_{j\in J}$. Then $f(-z_j) = -f(z_j) = a$, and we have $\{-z_j\}_{j\in J}$ an infinite number of solutions. Thus the only possible point where this may occur is $a = 0$. However, note that defining $x_n = 1/n\pi$ for $n \in \mathbb{Z}$ we have $f(x_n) = x_n\sin(1/x_n) = 0$, and taking n as big as we need, we have infinite $x_n \in \{z : 0 < |z| < \delta\}$ for arbitrarily small $\delta \in \mathbb{R}$. This means that $f(\{z : 0 < |z| < \delta\}) = \mathbb{C}$ for arbitrarily small $\delta \in \mathbb{R}$.

Suppose n is odd. A similar argument to the one presented in [2] justifying $n = 1$ can be used to prove that $\sin(z) = az^n$ has infinitely many roots (using that n is odd) for every complex number $a \in \mathbb{C}$, and thus $f(\{z : 0 < |z| < \delta\}) = \mathbb{C}$ for arbitrarily small $\delta \in \mathbb{R}$.

Problem 4 (p. 110)

Consider $f(z) = 1/z(z-1)(z-2)$, give the Laurent expansion in the cases:

1. $0 < |z| < 1$, notice that the term $1/z$ is already in a Laurent expansion form, so we just have to determine the corresponding to the other two. Now:

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{k=0}^{\infty} z^k,$$

because $|z| < 1$. Moreover:

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-z/2} = \frac{-1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k},$$

because $|z/2| < 1$. Then:

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\sum_{k=0}^{\infty} z^k\right)\frac{1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k} = \sum_{k=-1}^{\infty}(1-2^{-k-2})z^k.$$

2. $1 < |z| < 2$, notice that the series for $1/z$ and $1/(z-2)$ computed above are still valid. However, now:

$$\frac{1}{z-1} = \frac{-1}{z}\frac{1}{1-1/z} = \frac{-1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k},$$

because $|z| > 1$. Then:

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\frac{-1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k}\right)\frac{-1}{2}\sum_{k=0}^{\infty}\frac{z^k}{2^k} = -\sum_{k=-\infty}^{-2} z^k - \sum_{k=-1}^{\infty} 2^{-n-2}z^k.$$

3. $2 < |z| < \infty$, notice that the series for $1/z$ and $1/(z-1)$ computed above are still valid. However, now:

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-z/2} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{2^k}{z^k},$$

because $|z| > 2$. Then:

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{z}\left(\frac{-1}{z}\sum_{k=0}^{\infty}\frac{1}{z^k}\right)\frac{1}{z}\sum_{k=0}^{\infty}\frac{2^k}{z^k} = \sum_{k=-\infty}^{-2}(2^{-n-2}-1)z^k.$$

Problem 5 (p. 110)

Show that $f(z) = \tan(z)$ is analytic in $\mathbb{C}$ except for simple poles at $z_n = \pi/2 + n\pi$ for $n \in \mathbb{Z}$, and determine the singular part. We compute:

$$\lim_{z\to z_n} |\tan(z)| = \lim_{z\to z_n} \left|\frac{\sin(z)}{\cos(z)}\right| = \infty,$$

since the only points where $\cos(z) = 0$ is precisely when $z = z_n$ for $n \in \mathbb{Z}$, and $\sin(z_n) = 1$ for every $n \in \mathbb{Z}$. This proves that $f(z)$ is analytic in $\mathbb{C}$ except for the aforementioned poles. Moreover:

$$\lim_{z\to z_n} (z - z_n)\tan(z) = \lim_{z\to z_n} (z - z_n)\frac{\sin(z)}{\cos(z)} = \lim_{z\to z_n} \frac{(z - z_n)\cos(z) + \sin(z)}{-sin(z)} = -1,$$

by l'Hôpital's rule and thus:

$$\lim_{z\to z_n} (z - z_n)^2\tan(z) = \lim_{z\to z_n} (z - z_n) \lim_{z\to z_n} (z - z_n)\frac{\sin(z)}{\cos(z)} = 0,$$

meaning that the function $g_n(z) = (z - z_n)\tan(z)$ has a removable singularity at $z = z_n$ and defining $g_n(z_n) = -1$ this is an analytic function. This means that near (but not in) $z = z_n$ we have $f(z) = g_n(z)/(z - z_n)$ or by the above, the first term in the power series expansion near $z = z_n$ of $g_n(z)$ is -1, thus $-1/(z - z_n)$ is the singular part of $f(z)$ at $z = z_n$ for each $n \in \mathbb{Z}$.

Problem 7 (p. 110)

Let f have an isolated singularity at $z = a$ with $f(z) \neq 0$ as a function. Show that if either:

$$\lim_{z\to a} |z-a|^s |f(z)| = 0 \text{ or } \lim_{z\to a} |z-a|^s |f(z)| = \infty$$

for some $s \in \mathbb{R}$, then there is an integer $m \in \mathbb{Z}$ such that:

$$\lim_{z\to a} |z-a|^t |f(z)| = \begin{cases} 0 \text{ if } t > m, \\ \infty \text{ if } t < m. \end{cases}$$

We will do this in two parts:

1. Suppose $\lim_{z\to a} |z-a|^s |f(z)| = 0$, that is, $\lim_{z\to a} (z-a)^s f(z) = 0$, for some $s \in \mathbb{R}$. Then there is some $n \in \mathbb{N}^+$ with $\lim_{z\to a} (z-a)^n f(z) = 0$ and $\lim_{z\to a} (z-a)^{n+1} f(z) = 0$ (simply take $n > s$). This means that $(z-a)^n f(z)$ has a removable singularity at $z = a$ and we can extend it by defining 0 as the value at $z = a$. Then by [1, Corollary 3.9 (p. 79)] the function $(z-a)^n f(z)$ has a zero of finite order, say $k \in \mathbb{N}$, at $z = a$ and:
$$(z-a)^n f(z) = (z-a)^k h(z) \text{ with } h(a) \neq 0,$$
and h being analytic. Thus computing:
$$\lim_{z\to a} (z-a)^t f(z) = \lim_{z\to a} (z-a)^{t-n+k} h(z) = \begin{cases} 0 \text{ if } t > n-k, \\ \pm\infty \text{ if } t < n-k, \\ h(a) \neq 0 \text{ if } t = n-k. \end{cases}$$
Thus taking $m = n-k \in \mathbb{Z}$ is enough to prove what we wanted. Note that although not explicitly needed right now, when $t = m$ we obtain that the limit is non zero. This will be used in the next exercises with key effect.

2. Suppose $\lim_{z\to a} |z-a|^s |f(z)| = \infty$. Then there is some $n \in \mathbb{N}^+$ with $\lim_{z\to a} |z-a|^n |f(z)| = \infty$ (simply take $n < s$). This means that $(z-a)^n f(z)$ has a pole at $z = a$, say of order $l \in \mathbb{N}$. Then by [1, Proposition 1.4 (p. 105)] we can write:
$$(z-a)^n f(z) = h(z)/(z-a)^l \text{ with } h(a) \neq 0,$$
and h being analytic. Thus computing:
$$\lim_{z\to a} (z-a)^t f(z) = \lim_{z\to a} (z-a)^{t-n-l} h(z) = \begin{cases} 0 \text{ if } t > n+l, \\ \pm\infty \text{ if } t < n+l, \\ h(a) \neq 0 \text{ if } t = n+l. \end{cases}$$
Thus taking $m = n+l \in \mathbb{Z}$ is enough to prove what we wanted. Note that although not explicitly needed right now, when $t = m$ we obtain that the limit is non zero. This will be used in the next exercises with key effect.

Problem 8 (p. 110)

Let f, a and m as in the exercise above. Show:

1. $m = 0$ if and only if $z = a$ is a removable singularity and $f(a) \neq 0$.

   $\Rightarrow$) Suppose we have:

   $$\lim_{z\to a} |z-a|^s |f(z)| = \begin{cases} 0 \text{ if } s > 0, \\ \neq 0 \text{ if } s = 0. \end{cases} \implies \begin{cases} \lim_{z\to a}(z-a)f(z) = 0 \text{ since } s = 1, \\ \lim_{z\to a} f(z) \neq 0 \text{ since } s = 0. \end{cases}$$

   Hence by [1, Theorem 1.2 (p. 103)] we have a removable singularity and we can define $f(a) \neq 0$ to make it analytic.

   $\Leftarrow$) Suppose we have $\lim_{z\to a}(z-a)f(z) = 0$, then by Exercise 7 above there is $m \in \mathbb{Z}$ with:

   $$\lim_{z\to a} |z-a|^t |f(z)| = \begin{cases} 0 \text{ if } t > m, \\ \infty \text{ if } t < m. \end{cases}$$

   but since we have:

   $$\lim_{z\to a} |z-a|^t |f(z)| = \begin{cases} 0 \text{ if } t > 0, \\ \neq 0 \text{ if } t = 0. \end{cases}$$

   the only option is $m = 0$, as desired.

2. $m < 0$ if and only if $z = a$ is a removable singularity and f has a zero of order $-m$ at $z = a$.

   $\Rightarrow$) The condition that we know is that for $m < 0$ we have according to Exercise 7 above:

   $$\lim_{z\to a} |z-a|^t |f(z)| = \begin{cases} 0 \text{ if } t > m, \\ \infty \text{ if } t < m. \end{cases} \implies \begin{cases} \lim_{z\to a}(z-a)f(z) = 0 \text{ since } s = 1, \\ \lim_{z\to a} f(z) \neq 0 \text{ since } s = 0. \end{cases}$$

   Hence by [1, Theorem 1.2 (p. 103)] we have a removable singularity and we can define $f(a) = 0$ to make it analytic. Moreover, we have that:

   $$\begin{cases} \lim_{z\to a}(z-a)(z-a)^m f(z) f(z) = 0 \text{ since } s = m+1, \\ \lim_{z\to a}(z-a)^m f(z) \neq 0 \text{ since } s = m, \end{cases}$$

   This means that $(z-a)^{-m} f(z)$ has a removable singularity at $z = a$ (with $-m$ being the smallest since $\lim_{z\to a}(z-a)(z-a)^{m-1} f(z) \neq 0$), therefore f has a pole of order $-m$ at $z = a$, as desired.

   $\Leftarrow$) Let f have a removable singularity at $z = a$ of order $-m$, that is, we can write $f(z) = (z-a)^{-m} g(z)$ with $g(z)$ analytic, $g(a) \neq 0$. Hence we obtain as desired:

   $$\lim_{z\to a}(z-a)^t f(z) = \lim_{z\to a}(z-a)^{t-m} g(z) = \begin{cases} 0 \text{ if } t > m, \\ \pm\infty \text{ if } t < m, \\ g(a) \neq 0 \text{ if } t = m. \end{cases}$$

3. $m > 0$ if and only if $z = a$ is a pole of f of order $-m$.

   $\Rightarrow$) The condition that we know is that for $m > 0$ we have according to Exercise 7 above:

$$\lim_{z \to a} |z-a|^t |f(z)| = \begin{cases} 0 \text{ if } t > m, \\ \infty \text{ if } t < m. \end{cases} \implies \begin{cases} \lim_{z\to a} (z-a)(z-a)^m f(z) = 0 \text{ since } s = m+1, \\ \lim_{z\to a} (z-a)^m f(z) \neq 0 \text{ since } s = m, \\ \lim_{z\to a} (z-a)^{m-1} f(z) = \pm\infty \text{ since } s = m-1. \end{cases}$$

   Hence m is the smallest such that $(z-a)^m f(z)$ has a removable singularity at $z = a$, thus f has a pole of order m, as desired.

   $\Leftarrow$) If f has a pole of order m at $z = a$ we can write $h(z) = (z-a)^m f(z)$ an analytic function (it may have a removable singularity at $z = a$, we just have to define $h(a)$ properly). Then we obtain as desired:

$$\lim_{z \to a} |z-a|^t |f(z)| = \lim_{z \to a} |z-a|^{t-m} |h(z)| = \begin{cases} 0 \text{ if } t > m, \\ \infty \text{ if } t < m, \\ |h(a)| \neq 0 \text{ if } t = m. \end{cases}$$

Problem 9 (p. 110)

Prove that a function f has an essential singularity at $z = a$ if and only if neither $\lim_{z\to a} |z-a|^s |f(z)| = 0$ nor $\lim_{z\to a} |z-a|^s |f(z)| = \infty$ hold for any real number $s \in \mathbb{R}$. We will prove both directions by contrapositive.

$\Rightarrow$) If f is such that either $\lim_{z\to a} |z-a|^s |f(z)| = 0$ or $\lim_{z\to a} |z-a|^s |f(z)| = \infty$ hold for certain real number $s \in \mathbb{R}$, then there exists $m \in \mathbb{Z}$ with:

$$\lim_{z\to a} |z-a|^t |f(z)| = \begin{cases} 0 \text{ if } t > m, \\ \infty \text{ if } t < m, \\ \neq 0 \text{ if } t = m, \end{cases}$$

by Exercise 7. Then by Exercise 8 we conclude that the only options are that $z = a$ is a removable singularity or a pole, meaning that it can never be an essential singularity.

$\Leftarrow$) Assume f has a removable or a pole at $z = a$. Then by Exercise 8 there exists an $m \in \mathbb{Z}$ such that:

$$\lim_{z\to a} |z-a|^t |f(z)| = \begin{cases} 0 \text{ if } t > m, \\ \infty \text{ if } t < m, \end{cases}$$

as desired.

Problem 12 (p. 111)

For this exercise, we will use [1, 1.11 Laurent Series Development (p. 107)] to compute what we are asked.

1. Let $\lambda \in \mathbb{C}$, show that:

$$\exp(\lambda(z + 1/z)/2) = a_0 + \sum_{n=1}^{\infty} a_n(z^n + 1/z^n)$$

for $0 < |z| < \infty$ with $a_n = \int_0^\pi e^{\lambda\cos(nt)}\cos(nt)/\pi dt$ for $n \geq 0$. Using that $\exp(\lambda(z + 1/z)/2)$ is analytic for $0 < |z| < \infty$, integrating over $z = e^{it}$ (and $dz = ie^{it}dt$) for $-\pi \leq t \leq \pi$ (notice that $e^{it} + e^{-it} = 2\cos(t)$ and $e^{-int} = \cos(nt) - i\sin(nt)$) hence we have that:

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\lambda\cos(t)}}{e^{(n+1)it}} ie^{it}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{\lambda\cos(t)}}{e^{nit}}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda\cos(t)}\cos(nt)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda\cos(t)}\sin(nt)dt \\ &= \frac{1}{\pi}\int_{0}^{\pi} e^{\lambda\cos(t)}\cos(nt)dt \end{aligned}$$

where the last step is justified because for all $n \in \mathbb{Z}$ we have that $\cos(nt)$ is even and $\sin(nt)$ is odd (in the variable t), thus by symmetry the positive and negative parts of the first integral are the same (hence we can multiply by two first integral and just integrate the positive part), and by symmetry the positive and negative parts of the first integral are the same in value but with different signs, thus it amounts to zero. Moreover, by this symmetry argument, we have that $a_n = a_{-n}$, thus the sum can be reorganized as we are asked and the final result is precisely what we wanted.

2. Let $\lambda \in \mathbb{C}$, show that:

$$\exp(\lambda(z - 1/z)/2) = b_0 + \sum_{n=1}^{\infty} b_n(z^n + (-1)^n/z^n)$$

for $0 < |z| < \infty$ with $b_n = \int_0^\pi \cos(nt - \lambda\sin(nt))/\pi dt$ for $n \geq 0$. Using that $\exp(\lambda(z - 1/z)/2)$ is analytic for $0 < |z| < \infty$, integrating over $z = e^{it}$ (and $dz = ie^{it}dt$) for $-\pi \leq t \leq \pi$ (notice that $e^{it} - e^{-it} = -2i\sin(t)$ and $e^{-int} = \cos(nt) - i\sin(nt)$) hence we have that:

$$\begin{aligned} b_n &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{\lambda i\sin(t)}}{e^{(n+1)it}} ie^{it}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{\lambda i\sin(t)}}{e^{nit}}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda i\sin(t)}(\cos(nt) - i\sin(nt))dt, \end{aligned}$$

now using that $e^{\lambda i \sin(t)} = \cos(\lambda \sin(t)) + i \sin(\lambda \sin(t))$, we can multiply:

$$\begin{aligned}[\cos(\lambda \sin(t)) + i \sin(\lambda \sin(t))][\cos(nt) - i \sin(nt)] &= \cos(nt) sin(\lambda \sin(t)) \\ + \sin(nt) \sin(\lambda \sin(t)) + i \sin(\lambda \sin(t)) \cos(nt) &+ i \cos(\lambda \sin(t)) \sin(nt) \\ &= \cos(nt - \lambda \sin(t)) + i \sin(nt + \lambda \sin(t)),\end{aligned}$$

where the first term is even and the second is odd (again in the variable t). Thus:

$$\begin{aligned}b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(nt - \lambda \sin(t)) dt &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(nt - \lambda \sin(t)) dt \\ &= \frac{1}{\pi} \int_{0}^{\pi} \cos(nt - \lambda \sin(t)) dt,\end{aligned}$$

since the first term being even means we can again multiply by two and only integrate the positive part, and the second term being odd means the integral amounts to zero. This is the integral we wanted, we now only need to prove that $b_{-n} = (-1)^n b_n$. For this, notice that:

$$f(-1/z) = \exp\left(\frac{\lambda}{2}\left(\frac{-1}{z} - \frac{-1}{1/z}\right)\right) = \exp\left(\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right) = f(z),$$

then expanding this equality in the power series:

$$\sum_{n=-\infty}^{\infty} b_{-n}(-1)^{-n} z^n = \sum_{n=-\infty}^{\infty} b_n (-1)^n z^{-n} = f(-1/z) = f(z) = \sum_{n=-\infty}^{\infty} b_n z^n.$$

This means that equalizing the terms of the same power we directly obtain the desired $b_{-n} = (-1)^n b_n$, proving the result.

Problem 13 (p. 111)

Let $R > 0$ and $G = \{z : |z| > R\}$, a function $f : G \longrightarrow \mathbb{C}$ is said to have a removable singularity, a pole or an essential singularity at infinity if $f(z^{-1})$ has a removable singularity, a pole or an essential singularity at $z = 0$. If f has a pole at infinity, the order of the pole is said to be the order of the pole of $f(z^{-1})$ at $z = 0$.

1. Prove that an entire function has a removable singularity at infinity if and only if it is a constant.

   $\Leftarrow$) Let $f(z) = a \in \mathbb{C}$, then $f(z^{-1}) = a$ hence $\lim_{z\to 0} zf(z^{-1}) = 0$ meaning that $f(z^{-1})$ has a removable singularity at $z = 0$ hence f has a removable singularity at infinity.

   $\Rightarrow$) Let f have a removable singularity at infinity, that is $f(z^{-1})$ has a removable singularity at $z = 0$. This means that defining $g(0)$ accordingly we have $f(z^{-1})) = g(z)$ an analytic function outside $z = 0$, hence:
   $$\lim_{z\to\infty} f(z) = \lim_{z\to 0} f(z^{-1}) = g(0) \in \mathbb{C}$$
   hence f is bounded: we have that fixed $M \in \mathbb{R}$, we have $|f(z) - g(0)| < M$ for $|z| \geq R$ since f is continuous in $\overline{B(0, R)}$ compact, thus $f(z) \leq \max\{M, g(0)\}$ is bounded, and by Liouville's Theorem it is constant.

2. Prove that an entire function has a pole at infinity of order m if and only if it is a polynomial of degree m.

   $\Leftarrow$) Let $f(z) = a_m m + \cdots + a_0$, then $f(z^{-1}) = a_m/z^m + \cdots + a_0$, thus by [1, Corollary 1.18 (p. 109)] we have that $f(z^{-1})$ has a pole of order m at $z = 0$ and thus $f(z)$ has a pole of order m at infinity.

   $\Rightarrow$) Let f have a pole of order m at infinity, then $f(z^{-1})$ has a pole of order m at $z = 0$. Thus by [1, Corollary 1.18 (p. 109)] we have that $f(z^{-1}) = b_{-m}/z^m + \cdots + b_{-1}/z + g(z)$ with $g(z)$ analytic. However, since $f(z)$ is analytic, we must have that $f(z^{-1}) = \sum_{k=0}^{\infty} a_k/z^k$. Comparing term by term, we see that there cannot be terms $1/z^n$ for $n > m$ and that $g(z) = g(0) = a_0$ thus we must have $f(z) = a_m z^m + \cdots + a_0$, a polynomial of degree m.

3. Characterize those rational functions which have a removable singularity at infinity. We will work with fractions of polynomials $r(z) = p(z)/q(z)$ with $p(z) = p_k z^k + \cdots + p_0$ and $q(z) = q_l z^l + \cdots + q_0$. Now $r(z)$ has a removable singularity at infinity if and only if $r(z^{-1})$ has a removable singularity at $z = 0$, that is, $\lim_{z\to 0} zr(z^{-1}) = 0$. Now:
   $$\lim_{z\to 0} zr(z^{-1}) = \lim_{z\to 0} z\frac{p_k/z^k + \cdots + p_1/z + p_0}{q_l/z^l + \cdots + q_1/z + q_0},$$
   and since $z \to 0$ means $1/z \to \infty$, the dominant terms are the ones of highest degree (in absolute value). Hence this limit is zero if and only if:
   $$\lim_{z\to 0} \frac{zp_k/z^k}{q_l/z^l} = \lim_{z\to 0} \frac{p_k}{q_l}\frac{zz^l}{z^k} = 0 \iff l + 1 > k \iff l \geq k$$

since both $l, k \in \mathbb{N}$. That is, a rational function has a removable singularity at infinity if and only if the numerator has degree less or equal to the denominator's degree.

4. Characterize those rational functions which have a pole of order m at infinity. We will work with fractions of polynomials $r(z) = p(z)/q(z)$ with $p(z) = p_k z^k + \cdots + p_0$ and $q(z) = q_l z^l + \cdots + q_0$. Now $r(z)$ has a pole at infinity of order m if and only if $r(z^{-1})$ has a pole at infinity of order m, that is (by [1, Corollary 1.18 (p. 109)]):

$$r(z^{-1}) = \frac{a_{-m}}{z^m} + \cdots + \frac{a_{-1}}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which happens if and only if $r(z^{-1})$ behaves like a_{-m}/z^m when $z \to 0$ because, as before, $z \to 0$ means $1/z \to \infty$ and the dominant term is the one of degree m. Hence we need:

$$\frac{p_k/z^k}{q_l/z^l} \sim \frac{a_{-m}}{z^m} \iff \frac{z^l}{z^k} \sim \frac{1}{z^m} \iff s - k = -k \iff s + m = k.$$

That is, a rational function has a pole of order m at infinity if and only if the numerator has degree exactly m greater than the denominator's degree.

References

[1] J. B. Conway, *Functions of One Complex Variable I*, Springer-Verlag, 2000.

[2] A. I. Markushevich, R. A. Silverman, *Theory of Functions of a Complex Variable*, American Mathematical Society, 2005.