Differential Geometry I - Midterm

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Exercise 3

1. Let M be an n-dimensional manifold equipped with a nowhere vanishing n-form $w \in \Omega^n(M)$. If X is a vector field on M, show that there is a unique scalar field F so that $L_X w = Fw$.

   We will write $w = g dx^1 \wedge \cdots \wedge dx^n$ the general expression of a form $w \in \Omega^n(M)$, where $g : M \longrightarrow \mathbb{R}$ is a smooth function. We will write $X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$ the general expression of a vector field on M. We want to compute $L_X w$, and for this we note that $dw = 0$, for any given coordinate we have $\iota_X(dx^i) = dx^i(X) = f^i$ (by the definition of $\iota_X(\eta)$ as letting η act on X on the first component, which is the only component in the 1-forms) and $\iota_X(w) = g\iota_X(dx^1 \wedge \cdots \wedge dx^n)$. Moreover, we have that:

   $$\begin{aligned} \iota_X(dx^1 \wedge \cdots \wedge dx^n) &= (\iota_X(dx^1)) \wedge dx^2 \wedge \cdots \wedge dx^n - dx^1 \wedge (\iota_X(dx^2 \wedge \cdots \wedge dx^n)) \\ &= f^1 dx^2 \wedge \cdots \wedge dx^n - (f^2 dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n) \\ &+ dx^1 \wedge dx^2 (\iota_X(dx^3 \wedge \cdots \wedge dx^n)) = \cdots \\ &= \sum_{i=1}^n (-1)^{i-1} f^i dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n. \end{aligned}$$

   Finally, when we add all the above together:

   $$\begin{aligned} L_X w &= \iota_X(dw) + d(\iota_X(w)) = d(g\iota_X(dx^1 \wedge \cdots \wedge dx^n)) \\ &= \sum_{i=1}^n (-1)^{i-1} d(gf^i) dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial(gf^i)}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n \frac{\partial(gf^i)}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n = \frac{1}{g} \sum_{i=1}^n \frac{\partial(gf^i)}{\partial x^i} g dx^1 \wedge \cdots \wedge dx^n \\ &= \frac{1}{g} \sum_{i=1}^n \frac{\partial(gf^i)}{\partial x^i} w \Longrightarrow F = \frac{1}{g} \sum_{i=1}^n \frac{\partial(gf^i)}{\partial x^i}, \end{aligned}$$

   obtaining the vector field F we desired.

2. Show that for $M = \mathbb{R}^3$ and $w = dx \wedge dy \wedge dz$, the scalar field found above agrees with the divergence of X, where X with components X^x, X^y and X^z then $\text{div}(X) = \frac{\partial X^x}{\partial x} + \frac{\partial X^y}{\partial y} + \frac{\partial X^z}{\partial z}$.

   Note that in this case we have $n = 3$ and $x^1 = x$, $x^2 = y$, $x^3 = z$ with $f^1 = X^x$, $f^2 = X^y$, $f^3 = X^z$ and $g = 1$. That means:

   $$F = \frac{1}{1} \sum_{i=1}^3 \frac{\partial(1 \cdot f^i)}{\partial x^i} = \frac{\partial X^x}{\partial x} + \frac{\partial X^y}{\partial y} + \frac{\partial X^z}{\partial z} = \text{div}(X),$$

   as desired.

3. Find an expression for the scalar field found above for a vector field $X = X^r \frac{\partial X^r}{\partial r} + X^\theta \frac{\partial X^\theta}{\partial \theta}$ on $\mathbb{R}^2 \setminus \{0\}$ equipped with $w = rdr \wedge \theta$.

   In this case we have $n = 2$ and $x^1 = r$, $x^2 = \theta$ with $f^1 = X^r$, $f^2 = X^\theta$ and $g = r$. That means:

$$F = \frac{1}{r} \sum_{i=1}^{2} \frac{\partial (r \cdot f^i)}{\partial x^i} = \frac{1}{r} \left(X^r + r \frac{\partial X^r}{\partial r} + \frac{\partial r}{\partial \theta} X^\theta + r \frac{\partial X^\theta}{\partial \theta} \right) = \frac{X^r}{r} + \frac{\partial X^r}{\partial r} + \frac{\partial X^\theta}{\partial \theta},$$

   because $\partial r / \partial \theta = 0$. This is the expression we desired.