Differential Geometry I - Homework 1

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Exercise 2

We consider S^n with the smooth structure given by the stereographic projections P_N and P_S (N being the north pole and S the south pole). In particular, the problem implicitly tells us that the stereographic projections are $\mathcal{C}^\infty$ related. We want to show that the following charts are $\mathcal{C}^\infty$ related to the stereographic projections:

$$\begin{array}{cccc} f & : & S^n \cap \{x \in \mathbb{R}^{n+1} : x^i > 0\} & \longrightarrow & \mathbb{R}^n \\ & & (x_1, \ldots, x_{n+1}) & \longmapsto & (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}), \end{array}$$

$$\begin{array}{cccc} g & : & S^n \cap \{x \in \mathbb{R}^{n+1} : x^i < 0\} & \longrightarrow & \mathbb{R}^n \\ & & (x_1, \ldots, x_{n+1}) & \longmapsto & (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}). \end{array}$$

In virtue of the above, it is enough to prove that f_i is related to P_S and g_i is related to P_N. To do this, we find an explicit expression for the stereographic projections, their inverses and the inverses of the two functions above.

Starting with P_N, we consider the vector joining the north pole N with $(x_1, \ldots, x_{n+1})$ a generic point in $S^n \setminus \{N\}$, that is, $\vec{v} = (x_1, \ldots, x_n, x_{n+1} - 1)$. Now we add this to the north pole enough times so that we land in the plane $x_{n+1} = 0$, that is, we want $k \in \mathbb{R}$ such that $1 + k(x_{n+1} - 1) = 0$, otherwise said $k = 1/1 - x_{n+1}$. Hence:

$$\begin{array}{cccc} P_N & : & S^n \setminus \{N\} & \longrightarrow & \mathbb{R}^n \\ & & (x_1, \ldots, x_{n+1}) & \longmapsto & \left(\frac{x_1}{1-x_{n+1}}, \ldots, \frac{x_n}{1-x_{n+1}}\right). \end{array}$$

To find the inverse P_N^{-1}, we let $(X_1, \ldots, X_n) \in \mathbb{R}^n$ and $(x_1, \ldots, x_{n+1}) \in S^n \setminus \{N\}$ be our coordinates and we have to consider the system of equations given by:

$$X_1 = \frac{x_1}{1 - x_{n+1}}, \ldots, X_n = \frac{x_n}{1 - x_{n+1}}, x_1^2 + \cdots + x_{n+1}^2 = 1,$$

that is:

$$x_1 = X_1(1-x_{n+1}), \ldots, x_n = X_n(1-x_{n+1}), (X_1(1-x_{n+1}))^2+\cdots+(X_n(1-x_{n+1}))^2+x_{n+1}^2 = 1$$

hence there are two solutions for x_{n+1}:

$$x_{n+1} = -1 \text{ and thus } x_1 = \cdots = x_n = 0 \text{ or } x_{n+1} = \frac{X_1^2 + \cdots + X_n^2 - 1}{X_1^2 + \cdots + X_n^2 + 1},$$

resulting in:

$$\begin{array}{cccc} P_N^{-1} & : & \mathbb{R}^n & \longrightarrow & S^n \setminus \{N\} \\ & & (X_1, \ldots, X_n) & \longmapsto & \left(\frac{2X_1}{X_1^2+\cdots+X_n^2+1}, \ldots, \frac{2X_n}{X_1^2+\cdots+X_n^2+1}, \frac{X_1^2+\cdots+X_n^2-1}{X_1^2+\cdots+X_n^2+1}\right). \end{array}$$

Notice how component wise, the function is $\mathcal{C}^\infty$.

Continuing with P_S, we consider the vector joining the south pole S with $(x_1, \ldots, x_{n+1})$ a generic point in $S^n \setminus \{S\}$, that is, $\vec{v} = (x_1, \ldots, x_n, x_{n+1} + 1)$. Now we add this to the north pole enough times so that we land in the plane $x_{n+1}=0$, that is, we want $k\in\mathbb{R}$ such that $-1+k(x_{n+1}+1)=0$, otherwise said $k=1/1+x_{n+1}$. Hence:

$$\begin{array}{cccc} P_S & : & S^n\setminus\{S\} & \longrightarrow & \mathbb{R}^n \\ & & (x_1,\ldots,x_{n+1}) & \longmapsto & \left(\frac{x_1}{1+x_{n+1}},\ldots,\frac{x_n}{1+x_{n+1}}\right). \end{array}$$

To find the inverse P_S^{-1}, we let $(X_1,\ldots,X_n)\in\mathbb{R}^n$ and $(x_1,\ldots,x_{n+1})\in S^n\setminus\{N\}$ be our coordinates and we have to consider the system of equations given by:

$$X_1=\frac{x_1}{1+x_{n+1}},\ldots,X_n=\frac{x_n}{1+x_{n+1}},x_1^2+\cdots+x_{n+1}^2=1,$$

that is:

$$x_1=X_1(1+x_{n+1}),\ldots,x_n=X_n(1+x_{n+1}),(X_1(1+x_{n+1}))^2+\cdots+(X_n(1+x_{n+1}))^2+x_{n+1}^2=1$$

hence there are two solutions for x_{n+1}:

$$x_{n+1}=1 \text{ and thus } x_1=\cdots=x_n=0 \text{ or } x_{n+1}=-\frac{X_1^2+\cdots+X_n^2-1}{X_1^2+\cdots+X_n^2+1},$$

resulting in:

$$\begin{array}{cccc} P_S^{-1} & : & \mathbb{R}^n & \longrightarrow & S^n\setminus\{S\} \\ & & (X_1,\ldots,X_n) & \longmapsto & \left(\frac{2X_1}{X_1^2+\cdots+X_n^2+1},\ldots,\frac{2X_n}{X_1^2+\cdots+X_n^2+1},-\frac{X_1^2+\cdots+X_n^2-1}{X_1^2+\cdots+X_n^2+1}\right). \end{array}$$

Notice how component wise, the function is $\mathcal{C}^\infty$.

Finally, a computation of the inverses f_j^{-1} and g_j^{-1} is obtained from imposing that the norm of the coordinates is one, hence all the components are the identity except the j-th one (note how $\hat{X}_j$ denotes that there is no X_j component in $\mathbb{R}^n$):

$$\begin{array}{cccc} f_j^{-1} & : & \mathbb{R}^n & \longrightarrow & S^n\cap\{x\in\mathbb{R}^{n+1}:x^i>0\} \\ & & (X_1,\ldots,\hat{X}_j,\ldots,X_n) & \longmapsto & \left(X_1,\ldots,\sqrt{1-X_1^2-\cdots-X_n^2},\ldots,X_n\right), \end{array}$$

$$\begin{array}{cccc} g_j^{-1} & : & \mathbb{R}^n & \longrightarrow & S^n\cap\{x\in\mathbb{R}^{n+1}:x^i<0\} \\ & & (X_1,\ldots,\hat{X}_j,\ldots,X_n) & \longmapsto & \left(X_1,\ldots,\sqrt{1-X_1^2-\cdots-X_n^2},\ldots,X_n\right). \end{array}$$

Again, notice how in the square root the term X_j^2 is not included in the subtraction. Moreover, notice how component wise in their domain, the functions are $\mathcal{C}^\infty$. It is important to mention that we need here to restrict to the domain, since the assignment $(X_1,\ldots,\hat{X}_j,\ldots,X_n)\longmapsto\sqrt{1-X_1^2-\cdots-X_n^2}$ is not $\mathcal{C}^\infty$ in general: if $1=X_1^2+\cdots+X_n^2$ this is not continuously differentiable. However, since we must land in $S^n\cap\{x\in\mathbb{R}^{n+1}: x^i>0\}$ or $S^n\cap\{x\in\mathbb{R}^{n+1}:x^i<0\}$, we have that $X_1^2+\cdots+X_n^2>0$, resolving this issue.

Now we have that all the compositions $f_j\circ P_S^{-1}$, $g_j\circ P_N^{-1}$, $P_S\circ f_j^{-1}$ and $P_N\circ g_j^{-1}$ are component wise $\mathcal{C}^\infty$ since over the real numbers the composition of $\mathcal{C}^\infty$ functions is $\mathcal{C}^\infty$, as desired. We will have to restrict us to the case of $\mathcal{C}^\infty$ functions over the reals multiple times in the following.

Exercise 3

1. Prove that all $\mathcal{C}^\infty$ functions are continuous, and that the composition of $\mathcal{C}^\infty$ functions is $\mathcal{C}^\infty$.

   For the first part, given $f : M \longrightarrow N$ a $\mathcal{C}^\infty$ function between manifolds, we want to check that it is continuous. Let $V \subset N$ be an open set. Since we have a maximal atlas in N that gives the notion of smoothness, and the charts in such atlas cover N and are compatible, we can find a homeomorphism y such that (y, V) belongs to the atlas in N. Now, we know that by definition, for any chart (x, U) of M we have that $y \circ f \circ x^{-1} : x(U) \longrightarrow y(V)$ is a real valued $\mathcal{C}^\infty$ function, hence it is continuous. Moreover, since x and y are diffeomorphisms in their respective smooth structures, they are homeomorphisms hence continuous with continuous inverse. Since the composition of continuous functions between topological spaces are continuous, we obtain that $f = y^{-1} \circ (y \circ f \circ x^{-1}) \circ x : f^{-1}(V) \longrightarrow V$ is a continuous function. In particular, for every $V \subset N$ open, we have that $f^{-1}(V)$ is open, which is the definition of $f : M \longrightarrow N$ being continuous, as desired.

   For the second part, let $f : M \longrightarrow N$ and $g : N \longrightarrow T$ be $\mathcal{C}^\infty$ functions. Let (x, U), (y, V), (z, W) be any charts in M, N, T respectively. Then:

   $$x \circ (g \circ f) \circ z^{-1} = (x \circ g \circ y^{-1}) \circ (y \circ g \circ z^{-1}) : z(W) \longrightarrow y(V) \longrightarrow x(U),$$

   which is a $\mathcal{C}^\infty$ function in the real numbers since is is a composition of two $\mathcal{C}^\infty$ functions in the real numbers, in virtue of the definition of f and g being $\mathcal{C}^\infty$ functions between manifolds. Hence, this means precisely that $g \circ f : M \longrightarrow T$ is a $\mathcal{C}^\infty$ function between manifolds, as desired.

2. Prove that a function $f : M \longrightarrow N$ is $\mathcal{C}^\infty$ if and only if $g \circ f : M \longrightarrow \mathbb{R}$ is $\mathcal{C}^\infty$ for every $g : N \longrightarrow \mathbb{R}$ that is $\mathcal{C}^\infty$.

   $\Rightarrow$) Let $f : M \longrightarrow N$ and $g : N \longrightarrow \mathbb{R}$ be $\mathcal{C}^\infty$ functions. By the above, we know that the composition of $\mathcal{C}^\infty$ functions is a $\mathcal{C}^\infty$ function, hence $g \circ f : M \longrightarrow \mathbb{R}$ is a $\mathcal{C}^\infty$ function.

   $\Leftarrow$) Let x, U, (y, V) be any charts on M, N respectively. Denoting by $y_i : N \longrightarrow \mathbb{R}$ the i-th component of y, we obtain by hypothesis that $y_i \circ f : M \longrightarrow \mathbb{R}$ is a $\mathcal{C}^\infty$ function. Moreover, since x^{-1} is a $\mathcal{C}^\infty$ function because x being a chart in particular means that it is a diffeomorphism, by the point above we obtain that $y_i \circ f \circ x^{-1} : x(U) \longrightarrow y_i(V)$ is a real valued $\mathcal{C}^\infty$ function. Hence the function $y \circ f \circ x^{-1} : x(U) \longrightarrow y(V)$ is real valued and $\mathcal{C}^\infty$ component wise, meaning that it is $\mathcal{C}^\infty$ as a real valued function. This is precisely the definition of $f : M \longrightarrow N$ being a $\mathcal{C}^\infty$ function between manifolds.

Exercise 4

Let a function $f : \mathbb{H}^n \longrightarrow \mathbb{R}$ have two $\mathcal{C}^\infty$ extensions g and h defined on a neighborhood of $\mathbb{H}^n$. Prove that $\partial_j g$ coincides with $\partial_j h$ on $\mathbb{R}^{n-1} \times \{0\}$.

First, recall that g and h being extensions of f means that $g(\mathbb{H}^n) = f(\mathbb{H}^n) = h(\mathbb{H}^n)$. Now, applying the definition of a partial derivative for $1 \leq j \leq n-1$, we obtain that:

$$\begin{aligned}
\partial_j g(x_1, \ldots, x_{n-1}, 0) &= \lim_{x \to 0} \frac{g(x_1, \ldots, x_j + x, \ldots, x_{n-1}, 0) - g(x_1, \ldots, x_{n-1}, 0)}{x} \\
&= \lim_{x \to 0} \frac{f(x_1, \ldots, x_j + x, \ldots, x_{n-1}, 0) - f(x_1, \ldots, x_{n-1}, 0)}{x} \\
&= \lim_{x \to 0} \frac{h(x_1, \ldots, x_j + x, \ldots, x_{n-1}, 0) - h(x_1, \ldots, x_{n-1}, 0)}{x} \\
&= \partial_j h(x_1, \ldots, x_{n-1}, 0),
\end{aligned}$$

since both $(x_1, \ldots, x_j + x, \ldots, x_{n-1}, 0), (x_1, \ldots, x_{n-1}, 0) \in \mathbb{H}^n$. Notice that for the last component, since we assume that g and h are $\mathcal{C}^\infty$, the limit that we want to compute is well defined, hence it doesn't matter how we make $x \in \mathbb{R}$ go to 0 because all of the ways will coincide. Hence we can apply the above for the case $x > 0$, $j = n$ and obtain:

$$\begin{aligned}
\partial_n g(x_1, \ldots, x_{n-1}, 0) &= \lim_{x \to 0} \frac{g(x_1, \ldots, x_{n-1}, x) - g(x_1, \ldots, x_{n-1}, 0)}{x} \\
&= \lim_{x \to 0} \frac{f(x_1, \ldots, x_{n-1}, x) - f(x_1, \ldots, x_{n-1}, 0)}{x} \\
&= \lim_{x \to 0} \frac{h(x_1, \ldots, x_{n-1}, x) - h(x_1, \ldots, x_{n-1}, 0)}{x} \\
&= \partial_n h(x_1, \ldots, x_{n-1}, 0),
\end{aligned}$$

since again $(x_1, \ldots, x_{n-1}, x), (x_1, \ldots, x_{n-1}, 0) \in \mathbb{H}^n$ because of the choice for x. This proves the desired result.

Exercise 5

Consider two smooth structures over $\mathbb{R}$, the first $\mathcal{U}$ being the maximal atlas containing the identity and the second $\mathcal{V}$ the maximal atlas containing taking the third power.

1. Show that the homeomorphisms ϕ_0 and ϕ_1 are not $\mathcal{C}^\infty$ related. For this, note that $\phi_1^{-1}(x) = \sqrt[3]{x}$, hence:

$$\begin{array}{rcccc} \phi_0 \circ \phi_1^{-1} & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \sqrt[3]{x} \end{array} \quad \text{with} \quad \begin{array}{rcccc} (\phi_0 \circ \phi_1^{-1})' & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \frac{1}{2\sqrt[3]{x^2}} \end{array}$$

is a function that is not $\mathcal{C}^\infty$ (it is not even $\mathcal{C}^1$, the derivative is not continuous), hence ϕ_0 and ϕ_1 are not $\mathcal{C}^\infty$ related.

2. Equip $M = \mathbb{R}$ with $\mathcal{U}$ and $N = \mathbb{R}$ with $\mathcal{V}$, consider:

$$\begin{array}{rcccc} f & : & M & \longrightarrow & N \\ & & x & \longmapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcccc} g & : & N & \longrightarrow & M \\ & & x & \longmapsto & x \end{array}$$

we want to find which of these maps is smooth. Notice that g is not smooth since for the defining charts ϕ_0 of $\mathcal{U}$ and ϕ_1 of $\mathcal{V}$ we have:

$$\begin{array}{rcccc} \phi_0 \circ g \circ \phi_1^{-1} & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \sqrt[3]{x} \end{array}$$

which, as stated above, is not $\mathcal{C}^\infty$. For f, we can apply the same charts and find that:

$$\begin{array}{rcccc} \phi_1 \circ f \circ \phi_0^{-1} & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & x \end{array}$$

which is indeed $\mathcal{C}^\infty$. Now consider any charts x in M and y in N. Their defining property is that they are $\mathcal{C}^\infty$ related to ϕ_0 and ϕ_1 respectively, in particular, $\phi_0 \circ x^{-1}$ is $\mathcal{C}^\infty$ and $y \circ \phi_1^{-1}$ is $\mathcal{C}^\infty$. Now:

$$y \circ f \circ x^{-1} = (y \circ \phi_1^{-1}) \circ (\phi_1 \circ f \circ \phi_0^{-1}) \circ (\phi_0 \circ x^{-1})$$

is a real valued $\mathcal{C}^\infty$ function since it is the composition of three real valued $\mathcal{C}^\infty$ functions. Since we proved this for any charts, this is precisely the definition of $f : M \longrightarrow N$ being $\mathcal{C}^\infty$ between manifolds.

3. Show that a function $h : N \longrightarrow \mathbb{R}$ is smooth at $x = 0$ if and only if the Taylor series of h at 0 contains only terms of the form x^{3k} with $k \in \mathbb{N}$.

$\Rightarrow$) Let $h : N \longrightarrow \mathbb{R}$ be smooth at $x = 0$. This in particular means that:

$$\begin{array}{rcccc} \phi_0 \circ h \circ \phi_1^{-1} & : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & h\left(\sqrt[3]{x}\right) \end{array}$$

is smooth. Since $h : \mathbb{R} \longrightarrow \mathbb{R}$, we have the usual form for its Taylor series:

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n \text{ hence } h\left(\sqrt[3]{x}\right) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} \sqrt[3]{x^n}.$$

Notice that since this is a Taylor series, it converges at $x = 0$ and thus we may find the derivative at $x = 0$ by differentiating inside the sum. However checking element wise, $\sqrt[3]{x^n}$ is $\mathcal{C}^\infty$ when $n = 3k$ for $k \in \mathbb{N}$, but in the rest of the cases we find that $\sqrt[3]{x^n}$ differentiated $n \mod 3$ times is no longer $\mathcal{C}^1$, hence $\sqrt[3]{x^n}$ is not $\mathcal{C}^\infty$. Thus, for h to be smooth, we need to only have the terms of the form x^{3k} for $k \in \mathbb{N}$ in the Taylor expansion.

$\Leftarrow$) Suppose $h : N \longrightarrow \mathbb{R}$ has as a Taylor expansion near $x = 0$:

$$h(x) = \sum_{n=0}^{\infty} h_n x^{3n}.$$

Then near $x = 0$ we have:

$$\phi_0 \circ h \circ \phi_1^{-1}(x) = h\left(\sqrt[3]{x}\right) = \sum_{n=0}^{\infty} h_n x^n.$$

This allow us to well define $\left(\phi_0 \circ h \circ \phi_1^{-1}\right)^{(n)}(0) = n! h_n$ for $n \in \mathbb{N}$. Hence $\phi_0 \circ h \circ \phi_1^{-1}$ has derivatives of all orders at $x = 0$, that is, it is smooth at $x = 0$.