Differential Geometry I - Homework 2

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Exercise 1

1. Show that the spherical coordinates are $\mathcal{C}^\infty$ related to the identity in $\mathbb{R}^n$. The spherical coordinates are:

$$(r, \theta_1, \ldots, \theta_{n-1}) : (0, \infty) \times (0, \pi) \times \cdots \times (0, \pi) \times (0, 2\pi) \longrightarrow \mathbb{R}^n$$

given by:

$$\begin{aligned}
x_1 &= r \cos\theta_1, \\
x_2 &= r \sin\theta_1 \cos\theta_2, \\
&\vdots \\
x_{n-1} &= r \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1}, \\
x_n &= r \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1}.
\end{aligned}$$

For simplicity of notation, we will denote the spherical coordinates by ψ. It is enough to prove that ψ is $\mathcal{C}^\infty$ related to the identity on an open subset of $\mathbb{R}^n$. For this, we first compute ψ^{-1}. Note that squaring all the equations and solving for r we clearly have that $r = \sqrt{x_1^2 + \cdots + x_n^2}$, hence we simply have to use that $\sin(\arccos(\theta)) = \sqrt{1 - \theta^2}$ to solve the i-th equation for θ_i, $i = 1, \ldots, n-1$, obtaining that component-wise ψ^{-1} is given by:

$$\begin{aligned}
r &= \sqrt{x_1^2 + \cdots + x_n^2}, \\
\theta_1 &= \arccos\left(\frac{x_1}{\sqrt{x_1^2 + \cdots + x_n^2}}\right), \\
&\vdots \\
\theta_{n-1} &= \arccos\left(\frac{x_{n-1}}{\sqrt{x_{n-1}^2 + x_n^2}}\right).
\end{aligned}$$

Considering that the domain where the function $\arccos(\theta)$ is defined is $\theta \in [-1, 1]$, but we must have $r > 0$ (notice the importance of the strictly greater than 0 since this will allow us to take an open), this means that we cannot take $\theta = 0$. Hence we choose to take:

$$\psi^{-1} : (0, \infty) \times \cdots \times (0, \infty) \longrightarrow (0, \infty) \times (0, \pi/2) \times \cdots \times (0, \pi/2),$$

notice that $(0, \infty) \times \cdots \times (0, \infty)$ is an open subset of $\mathbb{R}^n$, and for $\{x_i\}_{i=1}^n$ in the domain we have that $0 < x_i / \sqrt{x_i^2 + \cdots + x_n^2} < 1$ for $i = 1, \ldots, n$ (notice again the importance of the strict inequalities, which means that the arccos never touches 1 and thus it is $\mathcal{C}^\infty$ in the domain). Hence the arccos lands in $(0, \pi/2)$, and $(0, \infty) \times (0, \pi/2) \times \cdots \times (0, \pi/2)$ is an open subset of $(0, \infty) \times (0, \pi) \times \cdots \times (0, \pi) \times (0, 2\pi)$.

Finally, both $\psi \circ \mathrm{id}_{\mathbb{R}^n}^{-1}$ and $\mathrm{id}_{\mathbb{R}^n} \circ \psi^{-1}$ are component-wise $\mathcal{C}^\infty$ functions in their domains (since they are composition of $\mathcal{C}^\infty$ in their domains, in particular notice how we chose the domains accordingly so that this is satisfied), hence ψ and $\mathrm{id}_{\mathbb{R}^n}$ are $\mathcal{C}^\infty$ related, as we wanted to prove.

2. Show that the spherical coordinates are $\mathcal{C}^\infty$ related to the stereographic projections. The spherical coordinates are:

$$(\theta_1, \dots, \theta_{n-1}) : (0, \pi) \times \cdots \times (0, \pi) \times (0, 2\pi) \longrightarrow S^{n-1} \subset \mathbb{R}^n$$

given by:

$$\begin{aligned} x_1 &= \cos\theta_1, \\ x_2 &= \sin\theta_1 \cos\theta_2, \\ &\vdots \\ x_{n-1} &= \sin\theta_1 \cdots \sin\theta_{n-2} \cos\theta_{n-1}, \\ x_n &= \sin\theta_1 \cdots \sin\theta_{n-2} \sin\theta_{n-1}. \end{aligned}$$

For simplicity of notation, we will denote the spherical coordinates by φ (is is simply taking $r = 1$ in ψ). In virtue of Exercise 1 in Homework 1, it is enough to prove that φ is $\mathcal{C}^\infty$ related to f_j^{-1} defined by the functions (notice how we would "expect" φ to be related directly to f_j, but this is impossible since the domain and target do not coincide):

$$\begin{array}{rccc} f & : & S^{n-1} \cap \{x \in \mathbb{R}^n : x^i > 0\} & \longrightarrow & \mathbb{R}^{n-1} \\ & & (x_1, \dots, x_n) & \longmapsto & (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), \end{array}$$

for $i = 1, \dots, n$ on an open subset of $\mathbb{R}^n$. Recall that:

$$\begin{array}{rccc} f_j^{-1} & : & \mathbb{R}^{n-1} & \longrightarrow & S^{n-1} \cap \{x \in \mathbb{R}^n : x^i > 0\} \\ & & (X_1, \dots, \hat{X}_j, \dots, X_{n-1}) & \longmapsto & \left(X_1, \dots, \sqrt{1 - X_1^2 - \cdots - X_{n-1}^2}, \dots, X_{n-1}\right). \end{array}$$

We know by the above that component-wise φ^{-1} is given by:

$$\begin{aligned} \theta_1 &= \arccos\left(\frac{x_1}{\sqrt{x_1^2 + \cdots + x_n^2}}\right), \\ &\vdots \\ \theta_{n-1} &= \arccos\left(\frac{x_{n-1}}{\sqrt{x_{n-1}^2 + x_n^2}}\right). \end{aligned}$$

and we choose to take:

$$\varphi^{-1} : (0, \infty) \times \cdots \times (0, \infty) \longrightarrow (0, \pi/2) \times \cdots \times (0, \pi/2),$$

where by the reasoning above we have that $(0, \infty) \times \cdots \times (0, \infty)$ is an open subset of $\mathbb{R}^{n-1}$ and $(0, \pi/2) \times \cdots \times (0, \pi/2)$ is an open subset of $(0, \pi) \times \cdots \times (0, \pi) \times (0, 2\pi)$.

Notice how $\varphi \circ f_j$ is well defined since $f_j(S^{n-1} \cap \{x \in \mathbb{R}^n : x^i > 0\}) \subset (0, 1) \times \cdots \times (0, 1) \subset (0, \pi) \times \cdots \times (0, \pi) \times (0, 2\pi)$, hence we can apply φ. This is a $\mathcal{C}^\infty$ function because component-wise it is a composition of $\mathcal{C}^\infty$ functions (as usual, in their domains).

Moreover, notice how we need to restrict the domain to make $f_j^{-1} \circ \varphi^{-1}$ well defined. To do this, we start by determining values for x_n, say $x_n \in (1, 2)$. Looking at the expression for θ_{n-1} and imposing $0 < \theta_{n-1} < 1$, we can find for each x_n an interval where x_{n-1} is defined. Since the equation determining this interval is continuous, we obtain an open rectangle where θ_{n-1} behaves as desired. Repeating this process (notice how we essentially use that we can solve the equation $\theta = \arccos\left(x/\sqrt{x^2 + y^2}\right)$ for x given y), we obtain an open domain $D \subset \mathbb{R}^{n-1}$ such that $\varphi^{-1}(D) \subset (0, 1) \times \cdots \times (0, 1)$, hence we can apply f_j^{-1}. This is a $\mathcal{C}^\infty$ function because component-wise it is a composition of $\mathcal{C}^\infty$ functions (again, in their domains).

Exercise 2

Let M_1, M_2 be smooth manifolds, we equip $M_1 \times M_2$ with the smooth structure given by product of charts: for (x_1, U_1) and (x_2, U_2) charts of M_1 and M_2 respectively, we determine that $x_1 \times x_2 : U_1 \times U_2 \longrightarrow \mathbb{R}^{m_1+m_2}$ defined by $x_1 \times x_2(p_1, p_2) = (x_1(p_1), x_2(p_2))$ are in our atlas. Notice how this automatically makes the projections $\pi_i : M_1 \times M_2 \longrightarrow M_i$ for $i = 1, 2$ smooth.

1. Show that $M_1 \times M_2$ is diffeomorphic to $M_2 \times M_1$. For this, consider the map $f : M_1 \times M_2 \longrightarrow M_2 \times M_1$ given by $f(p_1, p_2) = (p_2, p_1)$. This clearly bijective having as inverse $g : M_2 \times M_1 \longrightarrow M_1 \times M_2$ given by $g(p_2, p_1) = (p_1, p_2)$. Both f and g are obviously continuous, since given $U_1 \subset M_1$ and $U_2 \subset M_2$ opens, we have that for the basic opens: $f^{-1}(U_2 \times U_1) = U_1 \times U_2$ and $g^{-1}(U_1 \times U_2) = U_2 \times U_1$, both open in $M_1 \times M_2$ and $M_2 \times M_1$ respectively. Thus f is a homeomorphism. Now, consider (x_1, U_1), (x_2, U_2) and (x_2', U_2'), (x_1', U_1') determining charts of $M_1 \times M_2$ and $M_2 \times M_1$ respectively. For $(p_1, p_2) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ we have:

$$\begin{aligned} x_2' \times x_1' \circ f \circ (x_1 \times x_2)^{-1}(p_1, p_2) &= x_2' \times x_1' \circ f \circ x_1^{-1} \times x_2^{-1}(p_1, p_2) \\ &= x_2' \times x_1' \circ x_2^{-1} \times x_1^{-1}(p_2, p_1) \\ &= x_2' \circ x_2^{-1} \times x_1' \circ x_1^{-1}(p_2, p_1), \end{aligned}$$

$$\begin{aligned} x_1 \times x_2 \circ g \circ (x_2' \times x_1')^{-1}(p_2, p_1) &= x_1 \times x_2 \circ g \circ (x_2')^{-1} \times (x_1')^{-1}(p_2, p_1) \\ &= x_1 \times x_2 \circ (x_1')^{-1} \times (x_2')^{-1}(p_1, p_2) \\ &= x_1 \circ (x_1')^{-1} \times x_2 \circ (x_2')^{-1}(p_1, p_2), \end{aligned}$$

   where both are $\mathcal{C}^\infty$ since component-wise they are $\mathcal{C}^\infty$ since x_2', x_2 and x_1', x_1 are $\mathcal{C}^\infty$ related (they belong to the same atlas). This means that f and g are both smooth, thus f is a diffeomorphism, as desired.

2. Show that the slice maps are differentiable:

$$\begin{aligned} \varphi_1 \ : \ M_1 &\longrightarrow M_1 \times M_2 \\ q_1 &\longmapsto (q_1, \overline{q_2}) \end{aligned}$$

$$\begin{aligned} \varphi_2 \ : \ M_2 &\longrightarrow M_1 \times M_2 \\ q_2 &\longmapsto (\overline{q_1}, q_2) \end{aligned}$$

   where $\overline{q_2} \in M_2$ and $\overline{q_1} \in M_1$ are fixed. We have in the above notation:

$$x_1' \times x_2' \circ \varphi_1 \circ x_1^{-1}(p_1) = x_1' \times x_2'(x_1^{-1}(p_1), \overline{q_2}) = (x_1' \circ x_1^{-1}(p_1), x_2'(\overline{q_2})),$$

$$x_1' \times x_2' \circ \varphi_2 \circ x_2^{-1}(p_2) = x_1' \times x_2'(\overline{q_1}, x_2^{-1}(p_2)) = (x_1'(\overline{q_1}), x_2' \circ x_2^{-1}(p_2)),$$

   where both are $\mathcal{C}^\infty$ since component-wise they are $\mathcal{C}^\infty$ since x_2', x_2 and x_1', x_1 are $\mathcal{C}^\infty$ related (they belong to the same atlas) and $x_2'(\overline{q_2})$, $x_1'(\overline{q_1})$ are constant.

3. Show that a map $f : N \longrightarrow M_1 \times M_2$ is smooth if and only if $\pi_i \circ f : N \longrightarrow M_i$, $i = 1, 2$, are smooth.

$\Rightarrow$) As we noticed above, with the induced structure on $M_1 \times M_2$ the projections are smooth. Hence using Exercise 3 in Homework 1, the composition of smooth functions is smooth (and f is smooth by hypothesis), thus $\pi_i \circ f : N \longrightarrow M_i$, $i = 1, 2$, are smooth.

$\Leftarrow$) Use the above notation for charts of $M_1 \times M_2$, and let (y, U) a chart of N (say onto $\mathbb{R}^n$). Consider:

$$\begin{aligned} x_1 \times x_2 \circ f \circ y^{-1} &= x_1 \times x_2 \circ ([\pi_1 \circ f] \times [\pi_2 \circ f]) \circ y^{-1} \\ &= (x_1 \circ [\pi_1 \circ f] \times x_2 \circ [\pi_2 \circ f]) \circ y^{-1} \\ &= (x_1 \circ [\pi_1 \circ f] \circ y^{-1} \times x_2 \circ [\pi_2 \circ f] \circ y^{-1}), \end{aligned}$$

that is $\mathcal{C}^\infty$ since component-wise it is $\mathcal{C}^\infty$ since composition of $\mathcal{C}^\infty$ functions is $\mathcal{C}^\infty$ (we have x_1, x_2, y diffeomorphisms and $\pi_1 \circ f$, $\pi_2 \circ f$ smooth by hypothesis).

Exercise 3

Consider the coordinate system y given by (y^1, y^2) on $\mathbb{R}^2$ defined by $y^1 = a$, $y^2 = a + b$. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be smooth. Notice how y^{-1} is given by $(y^{-1})^1 = y^1$, $(y^{-1})^2 = y^2 - y^1$). This also coincides with the components of I^i in terms of y^j, as is to be expected.

1. We compute $\partial f/\partial y^1$ from the definition: given $p \in \mathbb{R}^2$, we have:

$$\begin{aligned} \frac{\partial f}{\partial y^1}(p) &= D_1(f \circ y^{-1})(y(p)) \\ &= D_1(f(p))D_1((y^{-1})^1(y(p))) + D_2(f(p))D_1((y^{-1})^2(y(p))) \\ &= \frac{\partial f}{\partial I^1}(p) - \frac{\partial f}{\partial I^2}(p), \end{aligned}$$

   since $D_1(f(p)) = \partial f/\partial I^1(p)$, $D_2(f(p)) = \partial f/\partial I^2(p)$, $D_1((y^{-1})^1) = 1$, $D_1((y^{-1})^2) = -1$ (the last two in virtue of the expressions for y^{-1}).

2. Applying the desired result, we obtain:

$$\frac{\partial f}{\partial y^1}(p) = \frac{\partial f}{\partial I^1}(p)\frac{\partial I^1}{\partial y^1}(p) + \frac{\partial f}{\partial I^2}(p)\frac{\partial I^2}{\partial y^1}(p) = \frac{\partial f}{\partial I^1}(p) - \frac{\partial f}{\partial I^2}(p),$$

   since clearly the differentiation of $(y^{-1})^1$ and $(y^{-1})^2$ with respect to y^1 yield 1 and -1 respectively. For a general f and p, we clearly have that $\partial f/\partial y^1(p) \neq \partial f/\partial I^1(p)$.

Exercise 4

Let $M(m,n)$ and $M(m,n;k)$ denote the $m \times n$ matrices and the subset of rank k.

1. Let X_0 have rank k, we want to prove the existence of permutation matrices P, Q so that PX_0Q has the top left $k \times k$ matrix non-singular. Notice that multiplying by Q having a 1 in (i,j) sends the column i to the column j, and multiplying by P having a 1 in (i,j) sends the row j to the column i.

   Since X_0 having rank k means that there are k linearly independent columns, letting $X_0 = (c_1 \dots c_n)$ be the form in columns, name them $c_{i_1}, \dots, c_{i_k}$. Using Q we can send c_{i_j} to the original position of c_j in X_0 for $j = 1, \dots, k$. Since these changes are well defined, there are $n-k$ columns left to arrange and there are $n-k$ spots left in Q to fill with 1, thus we can fill Q in such a way that it is a permutation matrix. Similarly, X_0 having rank k means X_0Q has rank k, thus there are k linearly independent rows, letting $X_0Q = (r_1 \dots r_m)^T$ be the form in rows, name them $r_{i_1}, \dots, r_{i_k}$. Using P we can send r_{i_j} to the original position of r_j in X_0Q for $j = 1, \dots, k$. Since these changes are well defined, there are $m-k$ rows left to arrange and there are $m-k$ spots left in P to fill with 1, thus we can fill P in such a way that it is a permutation matrix. Hence by this rearrangement, we have that PX_0Q has the first k rows and the first k columns linearly independent, and if we call this A_0, we have $A_0 \in M(k,k)$ with maximal rank, thus A_0 is non-singular.

2. We know that $\det(A_0) = d \neq 0$, say $d > 0$. We also know that the norm $||\cdot||_\infty$, that takes the maximum of the sums over the rows of the absolute values of the entries, is equivalent to the standard matrix norm (say the Euclidean norm $||\cdot||_2$). Since the determinant is a continuous function, for every $\delta > 0$ there is an $\epsilon' > 0$ such that if $||A - A_0||_\infty < \epsilon'$ then $|\det(A) - \det(A_0)| < \delta$. Choose $\delta = d/2$, then there is an $\epsilon' > 0$ such that $|\det(A) - \det(A_0)| < d/2$, that is, $0 \neq \det(A)$, in particular A is non-singular. Now the epsilon we want is $\epsilon = \epsilon'/k$, since having all the entries of $A - A_0$ less than ϵ'/k guarantees that $||A - A_0||_\infty < k\epsilon = \epsilon'$ and thus A is non-singular.

3. Prove that X has rank k if and only if $D = CA^{-1}B$.

   $\Leftarrow$) We are given that X has a decomposition with $A \in M(k,k)$ non-singular and $D = CA^{-1}B$. Multiplying by the non-singular matrix as hinted using $Y = -CA^{-1}$, we have that the rank is maintained. However, such multiplication yields $YA + C = 0$ and $YB + CA^{-1}B = 0$. Hence the rank of PXQ is the rank of the matrix $(AB) \in M(k,n)$, which is k since $A \in M(k,k)$ is non singular.

   $\Rightarrow$) Let PXQ have rank k, multiplying by the non-singular matrix as hinted using $Y = -CA^{-1}$, we have that the rank is maintained. Since $YA + C = 0$, the rank is maintained if and only if $YB + D = 0$ (notice that here we heavily use that $YB+D$ has zeroes at its left, hence having even one element non zero automatically increases the rank). This implies $D = -YB = CA^{-1}B$, as desired.

4. Consider the function:

$$\begin{array}{rccc} f & : & M(m,n) & \longrightarrow & M(m-k,n-k). \\ & & \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \longmapsto & CA^{-1}B - D \end{array}$$

Notice that it sends the bottom right $(m-k)\times(n-k)$ matrix to itself, hence somewhere in the partial differential matrix we will have a (negative) identity of size $(m-k)(n-k)$. This implies that the rank is equal or greater than the dimension of the image $M(m-k,n-k)$, in particular 0 is a regular value. This means in virtue of the result we are said to use that $f^{-1}(0)$ is an $mn-(m-k)(n-k) = k(m+n-k)$ dimensional submanifold of $M(m,n)$. Since $f^{-1}(0) = M(m,n;k)$ by the above, we obtain the desired result.

Exercise 5

Let M be a metric space, with $\mathcal{C}^\infty$ related homeomorphisms $x : U \longrightarrow \mathbb{R}^n$ covering M. We want to show that the n is invariant.

Since M is metric, there is a (continuous) distance function $d(\cdot,\cdot) : M \times M \longrightarrow \mathbb{R}$. Let $p, q \in M$ such that $p \in U$, $p \in V$ and $x : U \longrightarrow \mathbb{R}^n$, $y : V \longrightarrow \mathbb{R}^m$. If $d(p,q) = 0$, then $p = q$ and we may assume without loss of generality that $V \subset U$, thus x restricted to V is y and we have $n = m$. Suppose $d(p,q) = r > 0$. The function $d_p(\cdot) : M \longrightarrow [0,\infty)$ defined by $d_p(t) = d(p,t)$ is continuous because the distance is continuous. Consider $S = d_p^{-1}([0, r+\epsilon))$ for $\epsilon > 0$, since $[0, r+\epsilon)$ is open in $[0,\infty)$ we have that S is open in M. This means that there is a homeomorphism $z : S \longrightarrow \mathbb{R}^k$ that is $\mathcal{C}^\infty$ related to the ones we had (actually, we can build it from those). Moreover, $q \in V \cap S \neq \emptyset$ and $p \in U \cap S \neq \emptyset$, thus $\mathbb{R}^k \cong z(V \cap S) \cong y(V \cap S) \cong \mathbb{R}^m$ and $\mathbb{R}^k \cong z(U \cap S) \cong x(U \cap S) \cong \mathbb{R}^n$, meaning that $m = k = n$, as desired.