Differential Geometry I - Homework 3

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February 13th, 2017

Exercise 1

Let M an n dimensional manifold, $p \in M$. Consider $\mathcal{C}_p = \{\alpha : \mathbb{R} \longrightarrow M : \alpha(0) = p\}$. We say that two curves $\alpha_1, \alpha_2 \in \mathcal{C}_p$ are equivalent if for all charts (x, U) of M around p we have $(x \circ \alpha_1)'(0) = (x \circ \alpha_2)'(0)$. Denote by $\overline{\mathcal{C}}_p$ the space of equivalence classes of curves in $\mathcal{C}_p$.

1. $\overline{\mathcal{C}}_p$ can be made into a vector space. First, for $\alpha \in \mathcal{C}_p$, we will use the notation $\alpha'(0)$ for the equivalence class of α in $\overline{\mathcal{C}}_p$, that is, $\alpha'(0) = \overline{d(x \circ \alpha)(0)/dt}$. Moreover, notice that we can define a multiplication by scalars in $\mathcal{C}_p$ since for a curve $\alpha \in \mathcal{C}_p$ and $r \in \mathbb{R}$ we can set $r\alpha : \mathbb{R} \longrightarrow M$ given by $(r\alpha)(t) = \alpha(rt) = (\alpha \circ m_r)(t)$ where $m_r(t) = rt$, notice how $(r\alpha)(0) = p$. By the chain rule, notice how this means:

$$\frac{d(x \circ (r\alpha))}{dt}(0) = \frac{d(x \circ \alpha)}{dt}(m_r(0))\frac{dm_r}{dt}(0) = r\frac{d(x \circ \alpha)}{dt}(0) \Longrightarrow (r\alpha)'(0) = r\alpha'(0)$$

hence we define for $\alpha, \beta \in \mathcal{C}_p$ and $r \in \mathbb{R}$:

$$(\alpha + \beta)'(0) = \alpha'(0) + \beta'(0), \quad (r\alpha)'(0) = r\alpha'(0),$$

both of which are well defined since $\alpha'(0), \beta'(0) \in \mathbb{R}^n$, $r \in \mathbb{R}$. Now, we can readily check that equipped with these operations, $\overline{\mathcal{C}}_p$ satisfies all the axioms of $\mathbb{R}$ vector spaces:

   (a) For $\alpha, \beta, \gamma \in \mathcal{C}_p$ we have $\alpha'(0) + (\beta'(0) + \gamma'(0)) = (\alpha'(0) + \beta'(0)) + \gamma'(0)$.

   (b) For $\alpha, \beta \in \mathcal{C}_p$ we have $\alpha'(0) + \beta'(0) = \beta'(0) + \alpha'(0)$.

   (c) The map $p : \mathbb{R} \longrightarrow M$ given by $p(r) = p$ for every $r \in \mathbb{R}$ clearly belongs to $\mathcal{C}_p$. Moreover, for any chart (x, U) of M around p we have $(x \circ p)(r) = x(p)$ for every $r \in \mathbb{R}$, thus $x \circ p = 0$ as a function, hence $p'(0) = 0$. This means that for any $\alpha \in \mathcal{C}_p$ we have $\alpha'(0) + p'(0) = \alpha'(0) = p'(0) + \alpha'(0)$.

   (d) Given $\alpha \in \mathcal{C}_p$, we have $-\alpha'(0) = (-\alpha)'(0) \in \overline{\mathcal{C}}_p$, with $\alpha'(0) + (-\alpha)'(0) = p'(0) = (-\alpha)'(0) + \alpha'(0)$.

   (e) For $r, s \in \mathbb{R}$ and $\alpha \in \mathcal{C}_p$ we have $(r(s\alpha))'(0) = r(s\alpha)'(0) = rs\alpha'(0) = ((rs)\alpha)'(0)$.

   (f) Clearly $(1\alpha)'(0) = \alpha'(0)$.

   (g) For $r \in \mathbb{R}$ and $\alpha, \beta \in \mathcal{C}_p$ we have $(r(\alpha + \beta))'(0) = r(\alpha + \beta)'(0) = r\alpha'(0) + r\beta'(0) = (r\alpha)'(0) + (r\beta)'(0)$.

   (h) For $rs \in \mathbb{R}$ and $\alpha \in \mathcal{C}_p$ we have $((r+s)\alpha)'(0) = (r+s)\alpha'(0) = r\alpha'(0) + r\beta'(0) = (r\alpha)'(0) + (r\beta)'(0)$.

   Notice how although it may seem obvious from the definition, there is actually something to check, and the verification that the notation $\alpha'(0)$ for the class of $\alpha \in \mathcal{C}_p$ behaves well is key factor for the computations that we made.

2. Compute the dimension of $\overline{\mathcal{C}}_p$. For this, we will use the obvious map:

$$\begin{array}{cccc} \psi & : & \overline{\mathcal{C}}_p & \longrightarrow & \mathbb{R}^n \\ & & \alpha'(0) & \longmapsto & d(x \circ \alpha)(0)/dt \end{array}$$

where (x, U) is any chart of M around p. Notice that even if this may seem redundant and an even an abuse of notation, this is perfectly well defined: given $\alpha, \beta \in \mathcal{C}_p$ with $\alpha'(0) = \beta'(0)$, this means $d(x \circ \alpha)(0)/dt = d(x \circ \beta)(0)/dt$ by construction of $\overline{\mathcal{C}}_p$.

Clearly ψ is linear since the differentiation is linear (hence behaves well on the sums), and we already checked in the part above that by the chain rule the notation is well behaved with respect to multiplication by elements $r \in \mathbb{R}$.

Moreover, by this construction, ψ is clearly injective, since having for $\alpha, \beta \in \mathcal{C}_p$ that $d(x \circ \alpha)(0)/dt = d(x \circ \beta)(0)/dt$ for a chart (x, U) of M around p means by the compatibility conditions that this is true for all such charts, hence $\alpha'(0) = \beta'(0)$.

To prove surjectivity, let (x, U) be a chart of M around p, suppose $x(p) = (p_1, \ldots, p_n)$ since M has dimension n. Define:

$$\begin{array}{cccc} \alpha_i & : & \mathbb{R} & \longrightarrow & M \\ & & r & \longmapsto & x^{-1}(p_1, \ldots, p_i + r, \ldots, p_n) \end{array}$$

we now have that:

$$\begin{array}{cccc} x \circ \alpha_i & : & \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ & & r & \longmapsto & (p_1, \ldots, p_i + r, \ldots, p_n) \end{array}$$

and clearly $d(x \circ \alpha_i)(0)dt = (0, \ldots, 1, \ldots, 0)$ where the 1 is in the i-th component. Thus any element in $\mathbb{R}^n$ may be written as a linear combination of $d(x \circ \alpha_i)(0)dt$ which comes from a linear combination of $\alpha_i'(0)$ via ψ, meaning that ψ is surjective.

Hence $\mathcal{C}_p$ has the same dimension as $\mathbb{R}^n$, that is, n.

Exercise 2

With the notation as above, we consider $\mathcal{D}_p = \{l : \mathcal{C}^\infty(M) \longrightarrow \mathbb{R} : l \text{ is linear and } l(fg) = f(p)l(g) + g(p)l(f)\}$.

1. Show that the map $\mathcal{C}_p \longrightarrow \mathcal{D}_p$ given by $\alpha \longmapsto l_\alpha$ with for $f : M \longrightarrow \mathbb{R}$ smooth:

   $$l_\alpha(f) = \frac{d(f \circ \alpha)}{dt}(0)$$

   descends to a map $\overline{\mathcal{C}}_p \longrightarrow \mathcal{D}_p$ that is injective.

   We first observe that the map descends, that is, given a chart (x, U) of M around p and $\alpha, \beta \in \mathcal{C}_p$ with $\alpha'(0) = \beta'(0)$, we want to see that $l_\alpha = l_\beta$. Now for $f : M \longrightarrow \mathbb{R}$ smooth:

   $$\begin{aligned} l_\alpha(f) &= \frac{d(f \circ \alpha)}{dt}(0) = \frac{d(f \circ x^{-1} \circ x \circ \alpha)}{dt}(0) = (f \circ x^{-1})'(x \circ \alpha(0))\frac{d(x \circ \alpha)}{dt}(0) \\ &= (f \circ x^{-1})'(x \circ \alpha(0))\frac{d(x \circ \beta)}{dt}(0) = \frac{d(f \circ x^{-1} \circ x \circ \beta)}{dt}(0) = \frac{d(f \circ \beta)}{dt}(0) \\ &= l_\beta(f) \end{aligned}$$

   where we have used that $d(x \circ \alpha)(0)/dt = d(x \circ \beta)(0)/dt$.

   For injectivity, suppose we have $\alpha'(0), \beta'(0)$ with $l_\alpha = l_\beta$. This means that $l_\alpha(f) = l_\beta(f)$ for every $f \in \mathcal{C}^\infty(M)$. In particular taking $x_i : M \longrightarrow \mathbb{R}$ the i-th component of the chart (x, U), that is, $x_i = \pi_i \circ x$, we have that:

   $$\begin{aligned} l_\alpha(x_i) &= \frac{d(x_i \circ \alpha)}{dt}(0) = \frac{d(\pi_i \circ x \circ \alpha)}{dt}(0) = (\pi_i)'(x \circ \alpha(0))\frac{d(x \circ \alpha)}{dt}(0), \\ l_\beta(x_i) &= \frac{d(x_i \circ \beta)}{dt}(0) = \frac{d(\pi_i \circ x \circ \beta)}{dt}(0) = (\pi_i)'(x \circ \beta(0))\frac{d(x \circ \beta)}{dt}(0). \end{aligned}$$

   Now, notice that by the definition of π_i, we have that $(\pi_i)' = (0, \ldots, 1, \ldots, 0)$ where the 1 is in the i-th component. Hence on the right hand side of the equations above we have the i-th component of $d(x \circ \alpha)(0)/dt$ and $d(x \circ \beta)(0)/dt$. Since the equality holds for every $i = 1, \ldots, n$, this results in $d(x \circ \alpha)(0)/dt = d(x \circ \beta)(0)/dt$ hence $\alpha'(0) = \beta'(0)$, proving injectivity.

2. To show that this map is in fact an isomorphism, we will prove that it is linear. Notice how this suffices because by Theorem 3 on Page 79 of Spivak's coursebook, the space of derivations $\mathcal{D}_p$ has dimension n, and an injective morphism between vector spaces is automatically an isomorphism. Now, for $\alpha, \beta \in \mathcal{C}_p$, $r \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(M)$ we have that:

   $$\begin{aligned} l_(\alpha + \beta)(f) &= \frac{d(f \circ (\alpha + \beta))}{dt} = \frac{d(f \circ \alpha + f \circ \beta))}{dt} = \frac{d(f \circ \alpha)}{dt} + \frac{d(f \circ \beta))}{dt} \\ &= l_\alpha(f) + l_\beta(f) = (l_\alpha + l_\beta)(f) \end{aligned}$$

and:

$$\begin{aligned} l_{r\alpha}(f) &= \frac{d(f\circ(r\alpha))}{dt}(0) = \frac{d(f\circ x^{-1}\circ x\circ(r\alpha))}{dt}(0) \\ &= (f\circ x^{-1})'(x\circ(r\alpha)(0))\frac{d(x\circ(r\alpha))}{dt}(0) \\ &= (f\circ x^{-1})'(x\circ\alpha(0))r\frac{d(x\circ\alpha)}{dt}(0) = r\frac{d(f\circ x^{-1}\circ x\circ\alpha)}{dt}(0) = rl_\alpha(f) \end{aligned}$$

proving the desired linearity (notice how we used again verifications from the exercise above).

Exercise 3

Let $f : M \longrightarrow N$ be with the dimensions of M and N being n and m respectively.

1. Show that the map $f_* : \mathcal{C}_p \longrightarrow \mathcal{C}_{f(p)}$ given by $f_*(\alpha) = f \circ \alpha$ descends to a map $\overline{f}_* : \overline{\mathcal{C}}_p \longrightarrow \overline{\mathcal{C}}_{f(p)}$ that is linear. First, we show that it indeed descends, for $\alpha, \beta \in \mathcal{C}_p$ with $\alpha'(0) = \beta'(0)$ we have for any charts (x, U) of M around p and (y, V) of N around $f(p)$ that:

$$\begin{aligned}\frac{d(y \circ f \circ \alpha)}{dt}(0) &= \frac{d(y \circ f \circ x^{-1} \circ x \circ \alpha)}{dt} = (y \circ f \circ x^{-1})'(x \circ \alpha(0))\frac{d(x \circ \alpha)}{dt}(0) \\ &= (y \circ f \circ x^{-1})'(x \circ \alpha(0))\frac{d(x \circ \beta)}{dt}(0) = \frac{d(y \circ f \circ x^{-1} \circ x \circ \beta)}{dt} \\ &= \frac{d(y \circ f \circ \beta)}{dt}(0),\end{aligned}$$

hence $(f_*(\alpha))'(0) = (f_*(\beta))'(0)$ in $\overline{\mathcal{C}}_{f(p)}$, since clearly $f_*(\alpha)(0) = f(\alpha(0)) = f(p) = f(\beta(0)) = f_*(\beta)(0)$. For linearity, let $\alpha, \beta \in \mathcal{C}_p$, $r \in \mathbb{R}$, now:

$$\begin{aligned}\overline{f}_*(\alpha'(0) + \beta'(0)) &= \overline{f}_*((\alpha + \beta)'(0)) = f_*(\alpha + \beta)'(0) = (f \circ (\alpha + \beta))'(0) \\ &= (f \circ \alpha)'(0) + (f \circ \beta)'(0) = f_*(\alpha)'(0) + f_*(\beta)'(0) \\ &= \overline{f}_*(\alpha'(0)) + \overline{f}_*(\beta'(0))\end{aligned}$$

notice how we have used that we have proven in the exercises above that the classes in $\mathcal{C}_{f(p)}$ behave well under composition, addition and sums. Moreover:

$$\begin{aligned}\overline{f}_*((r\alpha)'(0)) &= f_*(r\alpha)'(0) = (f \circ (r\alpha))'(0) \\ r\overline{f}_*(\alpha'(0)) &= r(f \circ \alpha)'(0)\end{aligned}$$

where the right hand side of the last two equations is equal (as classes in $\mathcal{C}_{f(p)}$) since we have proven in the exercise above that $d(f \circ (r\alpha))(0)/dt = rd(f \circ \alpha)(0)/dt$. This is what we desired.

2. Show that the map $\mathcal{D}_p \longrightarrow \mathcal{D}_{f(p)}$ given by $f_*(l)(g) = l(g \circ f)$ is a linear transformation. For this, given $l, k \in \mathcal{D}_p$ and $g \in \mathcal{C}^\infty(N)$, we have:

$$f_*(l + k)(g) = (l + k)(g \circ f) = l(g \circ f) + k(g \circ f) = f_*(l)(g) + f_*(k)(g)$$

and:

$$f_*(rl)(g) = (rl)(g \circ g) = rl(g \circ f) = rf_*(l)(g),$$

hence the map is linear.

3. If $i_M : \overline{\mathcal{C}}_p \longrightarrow \mathcal{D}_p$, $i_N : \overline{\mathcal{C}}_{f(p)} \longrightarrow \mathcal{D}_{f(p)}$ are the isomorphisms in the previous problem, show that $f_* \circ i_M = i_N \circ \overline{f}_*$. For this, let $\alpha \in \mathcal{C}_p$, $g \in \mathcal{C}^\infty(N)$, we have:

$$\begin{aligned}(f_* \circ i_M)(\alpha'(0))(g) &= (f_* \circ l_\alpha)(g) = f_*(l_\alpha)(g) = l_\alpha(g \circ f) = \frac{d(g \circ f \circ \alpha)}{dt}(0) \\ (i_N \circ \overline{f}_*)(\alpha'(0))(g) &= (i_N((f \circ \alpha)'(0)))(g) = l_{f \circ \alpha}(g) = \frac{d(g \circ f \circ \alpha)}{dt}(0),\end{aligned}$$

which is an equality for any two input arguments, proving that $(f_* \circ i_M)(\alpha'(0)) = (i_N \circ \overline{f}_*)(\alpha'(0))$ as elements of $\mathcal{D}_{f(p)}$, hence $f_* \circ i_M = i_N \circ \overline{f}_*$ as functions.

Exercise 4

1. Let $i_V : V \longrightarrow V$ be the isomorphism given by $i_V(v)(\lambda) = \lambda(v)$ for $v \in V$. Show that for any linear transformation $f : V \longrightarrow W$ we have that $f^{**} \circ i_V = i_W \circ f$. To prove this, let $v \in V$ and $w \in W^*$, now:

$$\begin{aligned} (f^{**} \circ i_V)(v)(w) &= f^{**}(i_V(v))(w) = i_V(v)(f^*(w)) = f^*(w)(v) = w(f(v)) \\ (i_W \circ f)(v))(w) &= i_W(f(v))(w) = w(f(v)) \end{aligned}$$

   where we have simply used the definition of dual function. Since both right hand sides are the same, for any $v \in V$ and $w \in W^*$, we indeed have that $f^{**} \circ i_V = i_W \circ f$.

2. Show that there are no isomorphisms $i_V : V \longrightarrow V^*$ and $i_W : W \longrightarrow W^*$ so that $i_V = f^* \circ i_W \circ f$. Consider the case $V = W = \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(r) = 0$ for every $r \in \mathbb{R}$. Now $f* : W^* \longrightarrow V^*$ sends everything to the zero function by linearity: for every $\lambda \in W^*$ and $v \in V$ we have $f^*(\lambda)(v) = \lambda(f(v)) = \lambda(0) = 0$. This cannot be an isomorphism, hence i_V cannot be an isomorphism if we want the diagram to commute.

   This example is immediately generalized to arbitrary vector spaces by using as f the function that sends everything to zero.

Exercise 5

1. Consider the space $[0,1]\times\mathbb{R}^n$ identifying $(0,v)$ with $(1,T(v))$ where $T:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is an isomorphism of vector spaces. We show that this can be made into the total space of a vector bundle over S^1. For this, define:

$$\begin{array}{cccc} \pi & : & \frac{[0,1]\times\mathbb{R}^n}{(0,v)\sim(1,T(v))} & \longrightarrow & \frac{[0,1]}{0\sim 1} \\ & & (t,v) & \longmapsto & t \end{array}$$

which will be our projection map. That is, we have:

   (a) $E=\frac{[0,1]\times\mathbb{R}^n}{(0,v)\sim(1,T(v))}$, $B=S^1$.

   (b) $\pi:E\longrightarrow B$ as above.

   (c) $\pi^{-1}(t)=\{t\}\times\mathbb{R}^n$.

   (d) For $t\in S^1$, any open containint t must contain either $(t-\varepsilon,t+\varepsilon)\subset S^1$ if $0<t<1$ or $[0,\varepsilon)\cup(1-\varepsilon,1]\subset S^1$ if $t=0,1$, in both cases for $\varepsilon>0$ small enough. Hence we can define as charts the natural inclusion $(t-\varepsilon,t+\varepsilon)\subset\mathbb{R}$ in the first case and for the second:

$$\begin{array}{ccccc} \varphi & : & [0,\varepsilon)\cup(1-\varepsilon,1] & \longrightarrow & (-\varepsilon,\varepsilon)\subset\mathbb{R} \\ & & [0,\varepsilon)\ni t & \longmapsto & t \\ & & (1-\varepsilon,1]\ni t & \longmapsto & t-1 \end{array}$$

   notice how this is well defined (it is even an isomorphism onto its image) by the Gluing Lemma. Hence on E we have the induced charts which are the natural inclusion $(t-\varepsilon,t+\varepsilon)\times\mathbb{R}^n\subset\mathbb{R}\times\mathbb{R}^n$ in the first case and for the second:

$$\begin{array}{ccccc} \tilde{\varphi} & : & \frac{([0,\varepsilon)\cup(1-\varepsilon,1])\times\mathbb{R}^n}{(0,v)\sim(0,T(v))} & \longrightarrow & \mathbb{R}\times\mathbb{R}^n \\ & & \{1,0\}\times\mathbb{R}^n\ni(t,v) & \longmapsto & (0,v) \\ & & [0,\varepsilon)\times\mathbb{R}^n\ni(t,v) & \longmapsto & (t,v) \\ & & (1-\varepsilon,1]\times\mathbb{R}^n\ni(t,v) & \longmapsto & (t-1,v) \end{array}$$

   where we obviously have $\pi\circ\tilde{\varphi}=\varphi\circ\pi$.

2. Now, $\pi:E\longrightarrow B$ is orientable if and only if we can define a collection of orientations on each $\pi^{-1}(t)$ for $t\in S^1$. When $t\neq 0,1$ we have that everything is the trivial bundle, hence we have the standard orientation on each of the fibers (recall that the standard orientation is orientation preserving). For $\pi^{-1}(0)=\pi^{-1}(1)$ we have that the transformation T is compatible, by definition of E, with the standard orientation. However, this means that the collection of orientations is well defined if and only if T is orientation preserving, since otherwise we would have that the standard orientation, which is orientation preserving, would be compatible with an orientation reversing map, which is a contradiction.