Differential Geometry I - Homework 4

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Exercise 1

Let (x, U) and (y, V) be two charts on a manifold M, $p \in U \cap V$ and thus $y_* \circ (x_*)^{-1} : x(U \cap V) \times \mathbb{R}^n \longrightarrow y(U \cap V) \times \mathbb{R}^n$ is a smooth diffeomorphism. We want to show that its Jacobian matrix is:

$$D(y_* \circ (x_*)^{-1}) = \begin{pmatrix} D_j(y_*^i \circ (x_*)^{-1}) & 0 \\ X & D_j(y_*^i \circ (x_*)^{-1}) \end{pmatrix}.$$

For this, we will reason in each of the three relevant $n \times n$ square matrices that form $D(y_* \circ (x_*)^{-1})$:

1. Top left: first, notice how $(y_* \circ (x_*)^{-1})^i = y_*^i \circ (x_*)^{-1}$ since the only possible way to take components is when we land in $\mathbb{R}^n \times \mathbb{R}^n$. Hence since $y_* = \pi^{-1} \circ y \circ \pi$ we have that $y_*^i = y^i$ for the first $i = 1, \dots, n$ components, thus $(y_* \circ (x_*)^{-1})^i = y^i \circ x^{-1}$ since we are simply restricting us to the function from $x(U \cap V)$ to $y(U \cap V)$. Thus the corresponding matrix is simply the Jacobian of $y \circ x^{-1}$, that is, $(D_j(y^i \circ x^{-1}))_{i,j}$.

2. Top right: this is the $i \times j$ submatrix with $i < n$ and $j > n$, which shows the change in $y(U \cap V)$ given by changes in $\mathbb{R}^n$ maintaining $x(U \cap V)$ fixed. Notice how for a point $(u, v) \in x(U \cap V) \times \mathbb{R}^n$, changes in v the second n-tuple, that is, changes in $\mathbb{R}^n$, do not affect the base point in TM to which it corresponds: we are moving on the fiber, but the base point is determined by u hence remains the same. This means that the first n-tuple of $y(U \cap V) \times \mathbb{R}^n$, which controls the base points, is not affected by changes of v. This, by definition, means that the partial derivatives with respect to the second n-tuple are zero thus, as we desired, the matrix has all entries zero: $\partial((y_* \circ (x_*)^{-1})^i)/\partial I_j = 0$ for $i, j = 1, \dots, n$ (with the notation of $\{I_i\}_{i=1}^n$ for the coordinates of $\mathbb{R}^n$ in $x(U \cap V) \times \mathbb{R}^n$).

3. Bottom right: this part of the matrix is the restriction of $y_* \circ (x_*)^{-1}$ to the second n-tuple, that is, a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Hence what we have is an isomorphism from writing $\mathbb{R}^n$ in the coordinates given by x to writing $\mathbb{R}^n$ in the coordinates given by y, thus the Jacobian is just the corresponding change of basis matrix, that is, $(D_j(y^i \circ x^{-1}))_{i,j}$.

Exercise 2

Let M a manifold, $p \in M$ (say of dimension n). We denote $\mathcal{I}_p = \{f \in \mathcal{C}^\infty(M;\mathbb{R}) : f(p) = 0\}$ thus:

$$\mathcal{I}_p^2 = \left\{ \sum_{j=1}^k f_j g_j : f_j, g_j \in I_p \text{ for } i = 1, \ldots, k \right\}.$$

1. The map $i : \mathcal{I}_p \longrightarrow T_p^* M$ defined by $i(f)(v) = v(f)$ for $v \in T_p M$ is linear. For this, we just have to verify (abusing that a derivation is $\mathbb{R}$-linear) that for $f, g \in \mathcal{I}_p$ and $r \in \mathbb{R}$:

   (a) $i(f+g)(v) = v(f+g) = v(f) + v(g) = i(f)(v) + i(g)(v)$,

   (b) $i(rf)(v) = v(rf) = rv(f) = ri(f)(v)$.

2. Show that $\ker(i) = \mathcal{I}_p^2$, thus $\bar{i} : \mathcal{I}_p/\mathcal{I}_p^2 \longrightarrow T_p^* M$ is an injection. We prove both inclusions:

   $\subseteq$) Let $f \in \ker(i)$, that is, $0 = i(f)(v) = v(f)$ for every $v \in T_p M$. In particular, applying this to $\partial/\partial x_i, \ldots, \partial/\partial x_n$ a basis of $T_p M$, we obtain that to $\partial f(p)/\partial x_i = 0$ for every $i = 1, \ldots, n$. Consider now [1, Theorem A.58 (p.587)] the multivariate Taylor expansion of f near p given by a local chart (x, U), that is, for $q \in U$ we have:

   $$\begin{aligned} f(q) &= f(p) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(x_i(q) - x_i(p)) + \sum_{i=1}^n g_i(q)(x_i(q) - x_i(p)) \\ &= \sum_{i=1}^n g_i(q)(x_i(q) - x_i(p)), \end{aligned}$$

   with $g_1, \ldots, g_n \in \mathcal{C}^\infty(M;\mathbb{R})$ with $g_i(p) = 0$ for every $i = 1, \ldots, n$. Now, clearly $g_i \in \mathcal{I}_p$ and $(x(q) - x(p)) \in \mathcal{I}_p$, thus $f \in \mathcal{I}_p^2$ as desired.

   $\supseteq$) Let $f \in \mathcal{I}_p^2$, that is, $f = \sum_{j=1}^k f_j g_j$ with $f_j, g_j \in I_p$ for $j = 1, \ldots, k$. Now for any $v \in T_p M$:

   $$i(f)(v) = v(f) = v\left(\sum_{j=1}^k f_j g_j\right) = \sum_{j=1}^k v(f_j g_j) = \sum_{j=1}^k v(f_j) g_j(p) + f_j(p) v(g_j) = 0$$

   since $f_j(p) = 0 = g_j(p)$, thus $f \in \ker(i)$.

3. Show that $\bar{i} : \mathcal{I}_p/\mathcal{I}_p^2 \longrightarrow T_p^* M$ is an isomorphism. Given the above, we know that $\bar{i}$ is a morphism and an injection, hence we just have to prove that it is a surjection. For this, given a linear function $w : T_p M \longrightarrow \mathbb{R}$, it is enough to find an element $f \in \mathcal{I}_p$ such that $i(f)(v) = w(v)$ for every $v \in T_p M$ (since then taking the class $\overline{f}$

of f in $\mathcal{I}_p/\mathcal{I}_p^2$ we find that $\bar{i}(\overline{f}) = w$). Since for a chart (x, U) with $p \in U$ we have that $\partial/\partial x_i, \ldots, \partial/\partial x_n$ is a basis of T_pM, we have (and impose) that:

$$w^i = w\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f(p)}{\partial x_i} \text{ for } i = 1, \ldots, n$$

thus defining $f(q) = \sum_{i=1}^n w^i(x_i(q) - x_i(p))$ for $q \in U$ we obtain that $f(p) = 0$ and f is a smooth function with derivatives $\partial f(p)/\partial x_i = w^i$ for $i = 1, \ldots, n$. Thus $f \in \mathcal{I}_p$ and $i(f) = w$ as desired.

Exercise 3

Let $f : M^n \longrightarrow N^m$ be smooth and let (x, U), (y, V) be coordinate systems around p, $f(p)$ respectively.

1. For $g : N \longrightarrow \mathbb{R}$ we have that:

$$\begin{aligned}\frac{\partial(g \circ f)}{\partial x^i}(p) &= D_i(g \circ f \circ x^{-1})(x(p)) = D_i(g \circ y^{-1} \circ y \circ f \circ x^{-1})(x(p)) \\ &= \sum_{j=1}^{m} D_j(g \circ y)(y(f(p)))D_i((y \circ f \circ x^{-1})^j)(x(p)) \\ &= \sum_{j=1}^{m} D_j(g \circ y)(y(f(p)))D_i(y^j \circ f \circ x^{-1})(x(p)) \\ &= \sum_{j=1}^{m} \frac{\partial g}{\partial y^j}(f(p))\frac{\partial(y_j \circ f)}{\partial x^i}(p).\end{aligned}$$

2. We have for $g : N \longrightarrow \mathbb{R}$ that:

$$f_*\left(\frac{\partial}{\partial x^i}|_p\right)(g) = \frac{\partial(g \circ f)}{\partial x^i}(p) = \sum_{j=1}^{m} \frac{\partial(y_j \circ f)}{\partial x^i}(p)\frac{\partial g}{\partial y^j}(f(p))$$

   hence:

$$f_*\left(\frac{\partial}{\partial x^i}|_p\right) = \sum_{j=1}^{m} \frac{\partial(y_j \circ f)}{\partial x^i}(p)\frac{\partial}{\partial y^j}|_{f(p)}.$$

3. Using [1, Lemma 6.12 (p. 137)] and the differential expansion we have that:

$$(f^*dy^j)(p) = d(y^j \circ f)(p) = \sum_{i=1}^{n} \frac{\partial(y^j \circ f)}{\partial x^i}(p)dx^i(p).$$

4. We express the following in terms of dx^i for $i = 1, \dots, n$. First, recall that the pullback is a linear function. Moreover, it behaves multiplicatively with respect to finitely many tensor products since for two vectors $u, v \in T_pM$:

$$\begin{aligned}f^*(dy^i \otimes dy^j)(u, v) &= dy^i \otimes dy^j(f_*u, f_*v) = dy^i(f_*u)dy^j(f_*v) \\ &= f^*(dy^i)(u)f^*(dy^j)(v) = f^*(dy^i) \otimes f^*(dy^j)(u, v),\end{aligned}$$

so $f^*(dy^i \otimes dy^j) = f^*(dy^i) \otimes f^*(dy^j)$. Thus applying the above (always at p):

$$\begin{aligned}
f \quad ^* \left(\sum_{j_1,\ldots,j_k} a_{j_1} \cdots a_{j_k} dy^{j_1} \otimes \cdots \otimes dy^{j_k} \right) &= \sum_{j_1,\ldots,j_k} a_{j_1} \cdots a_{j_k} f^*(dy^{j_1} \otimes \cdots \otimes dy^{j_k}) \\
&= \sum_{j_1,\ldots,j_k} a_{j_1} \cdots a_{j_k} f^*(dy^{j_1}) \otimes \cdots \otimes f^*(dy^{j_k}) \\
&= \sum_{j_1,\ldots,j_k} a_{j_1} \cdots a_{j_k} \left(\sum_{i=1}^{n} \frac{\partial(y^{j_1} \circ f)}{\partial x^i}(p) dx^i \right) \otimes \cdots \otimes \left(\sum_{i=1}^{n} \frac{\partial(y^{j_k} \circ f)}{\partial x^i}(p) dx^i \right) \\
&= \sum_{j_1,\ldots,j_k} a_{j_1} \cdots a_{j_k} \left(\sum_{i_i,\ldots,i_n} \frac{\partial(y^{j_{i_1}} \circ f)}{\partial x^{i_1}}(p) \cdots \frac{\partial(y^{j_{i_n}} \circ f)}{\partial x^{i_n}}(p) dx^{i_1} \otimes \cdots \otimes dx^{i_n} \right)
\end{aligned}$$

where we sum over $i_1, \ldots, i_n \in \{i, \ldots, n\}$.

Exercise 4

Show that there exists a Riemannian metric on every manifold M (let n be its dimension). That is, we want a positive definite inner product g that sends two smooth vector fields X, Y to $g_p(X(p), Y(p))$ in a smooth way over M. The metric conditions means that for each $p \in M$ we want to have $g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ such that:

1. $g_{(p)}(u, v) = g_{(p)}(v, u)$ for every $u, v \in T_pM$,

2. $g_{(p)}(u, u) \geq 0$ for every $u \in T_pM$ and $g_{(p)}(u, u) = 0$ if and only if $u = 0$.

First, we notice that both the smooth conditions and the metric conditions are local, hence we can work with charts $\{(x_i, U_i)\}_{i\in I}$ of M. The complete hypothesis of M being a manifold yield that we can choose $\{U_i\}_{i\in I}$ to be a countable and locally finite cover of M, hence we have partitions of unity, that is, for $i \in I$ functions $\rho_i : M \longrightarrow [0, 1]$ with the support of ρ_i inside U_i: we have $\sum_{i\in I} \rho_i(p) = 1$ (in particular for every $p \in M$ the above sum is finite because the cover is locally finite)

Now, let $p \in M$ and $u, v \in T_pM$, let (x_i, U_i) be a chart covering p, we have that $(x_i)_* : T_pM \longrightarrow T_{x(p)}\mathbb{R}^n \cong \mathbb{R}^n$. We define $g_i(u, v) = \langle (x_i)_*(u), (x_i)_*(v)\rangle_{\mathbb{R}^n}$ where $\langle \cdot, \cdot\rangle_{\mathbb{R}^n}$ is the usual inner product in $\mathbb{R}^n$. The compatibility of the charts means that if (x_j, U_j) also covers p, then $g_j(u, v) = g_i(u, v)$ and this is well defined. Now, setting $g_i(u, v) = 0$ for $u, v \in T_qM$ with $q \notin U_i$ we obtain that g_i is defined for every $p \in M$. Now, the function $\rho_i g_i : T_pM \times T_pM \longrightarrow \mathbb{R}$ is well defined for every $p \in M$, it is smooth since ρ_i smoothens the behaviour of g_i in its support so that the transition to the value 0 is smooth, it is symmetric and bilinear as a consequence of $\langle \cdot, \cdot\rangle_{\mathbb{R}^n}$ being symmetric and bilinear, and finally it is positive definite when $p \in M$ belongs to the support of ρ_i by definition.

Define $g = \sum_{i\in I} \rho_i g_i$, where at each $p \in M$ we have a finite sum. Since a sum is smooth, and each of the summands is smooth, we obtain that g is a smooth function. In fact, since all the components are symmetric and bilinear, g inherits this properties and is also symmetric and bilinear. We only have to check that g is positive definite: for every $p \in M$ we have that since $\{\rho_i\}_{i\in I}$ are partitions of unity, there is a ρ_j with $\rho_j(p) \neq 0$, thus $p \in U_j$. Now for $u \in T_pM$ non zero, we have that $g(u, u) = \sum_{i\in I} \rho_i(p)g_i(u, u) \geq \rho_j(p)g_j(u, u) > 0$, and clearly $g_i(0, 0) = 0$ for all $i \in I$. Hence g is positive definite, and hence it is a Riemannian metric.

Exercise 5

Let $C, S \subset \mathbb{R}^n 2$ be respectively the unit circle and the boundary of the square of side 1 centered at the origin. We want to show that there is a homeomorphism $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $F(C) = S$, but there is no diffeomorphism with this property.

We will use the fact that in $\mathbb{R}^2$ the euclidean norm $||\cdot||_2$ and the infinite norm $||\cdot||_\infty$ are equivalent (this follows from relatively basic topology since in a finite dimensional vector space all norms are equivalent). Hence both as vector spaces and as topological spaces we have that $(\mathbb{R}^2, ||\cdot||_2) = (\mathbb{R}^2, ||\cdot||_\infty)$, where this is not only an isomorphism, but a real equality. Consider now:

$$\begin{array}{cccc} F & : & (\mathbb{R}^2, ||\cdot||_2) & \longrightarrow & (\mathbb{R}^2, ||\cdot||_\infty) \\ & & x & \longmapsto & \frac{||x||_2}{||x||_\infty} x \\ & & 0 & \longmapsto & 0 \end{array}, \qquad \begin{array}{cccc} F^{-1} & : & (\mathbb{R}^2, ||\cdot||_\infty) & \longrightarrow & (\mathbb{R}^2, ||\cdot||_2) \\ & & x & \longmapsto & \frac{||x||_\infty}{||x||_2} x \\ & & 0 & \longmapsto & 0 \end{array}$$

we have that F is continuous because the norms are continuous (as a consequence of the norms being equivalent), and for the same reason F^{-1} is continuous, and clearly they are inverses of each other. Now, by construction of F, we have that balls in $(\mathbb{R}^2, ||\cdot||_2)$, which are the usual open discs of radius $r > 0$, are sent to balls in $(\mathbb{R}^2, ||\cdot||_\infty)$, which are squares of side of length $r > 0$. This follows immediately from the way norms behave, but if this is not convincing enough, we can simply notice that given any point $x \in C$, it has $||x||_2 = 1$ thus $||F(x)||_\infty = ||x||_2 = 1$ and $F(x) \in S$. Moreover, this is not only injective since we can divide by the non-zero norms, but also surjective since using F^{-1} we can find for any $y \in S$ a point $x = F^{-1}(y) \in C$ such that $F(x) = y$. Hence $F(C) = S$ by a homeomorphism.

To prove that there is no diffeomorphism with this property, we assume there is one, say F, and proceed by contradiction. Note that since C has a structure of smooth manifold, we can restrict $F|_C$ and expect the image to be a smooth manifold. However, in the usual coordinates that we are using here, S cannot have a structure of smooth manifold: since C is a 1-dimensional manifold and $F|C$ is a diffeomorphism, S would have to be a 1-dimensional manifold; consider then the tangent space $TS = \coprod_{(\phi,U)\in\mathcal{A}(S)} T\phi(U)/\sim$ where $(x, v) \in T\phi(U)$ is equivalent to $(y, w) \in T\psi(V)$ if and only if $x = \phi(\psi^{-1}(y))$ and $v = D(\phi \circ \psi^{-1})_y(w)$. Note that (ϕ, U) are the usual charts in $\mathbb{R}^2$ because they are inherited from the usual charts from C by the diffeomorphism F. In particular at the point $(1, 1) \in S$ this means that the composition $\phi \circ \psi^{-1}$ is not smooth. Hence $T_{(1,1)}S$ is a single point since any (x, v), (y, w) with $x = (1, 1) = y$ (the interpretation of $x = \phi(\psi^{-1}(y))$ in $(1, 1)$) and everything is related to a single point. This is 0-dimensional, a contradiction with the fact that $T_{F(x)}S$ should be 1-dimensional for every $x \in C$, and implying that TS not only is not well defined, but since this definition is what we used to construct it, TS actually does not exist.

The contradiction comes from the fact that we supposed we had $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ diffeomorphism with $F(C) = S$, hence such a diffeomorphism cannot exist, as desired.

References

[1] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2003.