Differential Geometry I - Homework 5

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Exercise 1

1. Let $f : M \longrightarrow \mathbb{R}$ with $p \in M$ a critical point and M of dimension n. Given $v, w \in T_pM$, let X and Y be vector fields with $X(p) = v$ and $Y(p) = v$, that is, we can write:

$$v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}, \quad w = \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i}, \quad X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i}$$

with (x, U) a chart around $p \in M$ and $a^i, b^i : M \longrightarrow \mathbb{R}$ smooth functions with $a^i(p) = v^i$, $b^i(p) = w^i$ for $1 \leq i \leq n$. We define $f_{**}(v, w) = X(Y(f))(p)$, we want to show that since $[X, Y]_P(f) = 0$ (because the crossed derivatives are zero since p is a critical point of f, the way to see this is exactly what we will do in the following) we have that $f_{**}(v, w)$ is symmetric and well defined. First, using that the commutator is zero, we clearly have:

$$f_{**}(v, w) = X(Y(f))(p) = Y(X(f))(p) = f_{**}(w, v)$$

thus $f_{**}(v, w)$ is symmetric. Moreover, suppose we have $\tilde{X}$ and $\tilde{Y}$ be vector fields with $\tilde{X}(p) = v$ and $\tilde{Y}(p) = v$, that is, we can write:

$$\tilde{X} = \sum_{i=1}^{n} \tilde{a}^i \frac{\partial}{\partial x^i}, \quad \tilde{Y} = \sum_{i=1}^{n} \tilde{b}^i \frac{\partial}{\partial x^i}$$

with $\tilde{a}^i, \tilde{b}^i : M \longrightarrow \mathbb{R}$ smooth functions with $\tilde{a}^i(p) = v^i$, $\tilde{b}^i(p) = w^i$ for $1 \leq i \leq n$. Now:

$$\begin{aligned} X(Y(f))(p) &= \sum_{i=1}^{n} a^i(p) \frac{\partial}{\partial x^i} \left(\sum_{j=1}^{n} b^j \frac{\partial f}{\partial x^j} \right)(p) \\ &= \sum_{i=1}^{n} a^i(p) \left(\sum_{j=1}^{n} \left(\frac{\partial b^j}{\partial x^i} \frac{\partial f}{\partial x^j} + b^j \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \right)(p) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} a^i(p) b^j(p) \frac{\partial^2 f(p)}{\partial x^i \partial x^j} = \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j \frac{\partial^2 f(p)}{\partial x^i \partial x^j} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{a}^i(p) \tilde{b}^j(p) \frac{\partial^2 f(p)}{\partial x^i \partial x^j} = \tilde{X}(\tilde{Y}(f))(p) \end{aligned}$$

since p being a critical point of f means that $\partial f(p)/\partial x^i = 0$ for $1 \leq i \leq n$, so the crossed derivatives vanish. Thus $f_{**}(v, w)$ is well defined.

2. We show that (note that we have changed a bit the notation of the problem to better match what we want to illustrate). Let X and Y be as above, we have:

$$f_{**} \left(\sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \sum_{i=1}^{n} w^i \frac{\partial}{\partial x^i} \right) = X(Y(f))(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} v^i w^j \frac{\partial^2 f(p)}{\partial x^i \partial x^j}$$

in virtue of the computed above. This is what we wanted to prove.

3. Show that the rank of the matrix $\partial^2 f(p)/\partial x^i \partial x^j$ is independent of the coordinate system. For this, suppose we have another chart (y, V), then we may write $x(y)$ a change of variables and:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^l \partial y^k} &= \frac{\partial}{\partial y^l}\left(\sum_{j=1}^n \frac{\partial x^j}{\partial y^k}\frac{\partial f}{\partial x^j}\right) = \sum_{i=1}^n \frac{\partial x^i}{\partial y^l}\frac{\partial}{\partial x^i}\left(\sum_{j=1}^n \frac{\partial x^j}{\partial y^k}\frac{\partial f}{\partial x^j}\right) \\ &= \sum_{i=1}^n \frac{\partial x^i}{\partial y^l}\left(\sum_{j=1}^n \frac{\partial^2 x^j}{\partial x^i \partial y^k}\frac{\partial f}{\partial x^j} + \frac{\partial x^j}{\partial y^k}\frac{\partial^2 f}{\partial x^i \partial x^j}\right)\end{aligned}$$

which evaluated at our particular $p \in M$ having $\partial f(p)/\partial x^i = 0$ for $1 \leq i \leq n$ yields:

$$\frac{\partial^2 f(p)}{\partial y^l \partial y^k} = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial x^i(p)}{\partial y^l}\frac{\partial x^j(p)}{\partial y^k}\frac{\partial^2 f(p)}{\partial x^i \partial x^j}$$

which means that we can go between $\partial^2 f(p)/\partial x^i \partial x^j$ and $\partial^2 f(p)/\partial y^l \partial y^k$ by multiplication of the change of basis matrices between x and y. Since a change of basis matrix is invertible, it preserves the rank, hence the two matrices above have the same rank. Thus the rank of $\partial^2 f(p)/\partial x^i \partial x^j$ is independent of the chart we choose.

4. Let $f : M \longrightarrow N$ and $p \in M$ be a critical point of f. For $v, w \in T_pM$, X, Y vector fields over M as above and $g : N \longrightarrow \mathbb{R}$ smooth, we define $f_{**}(v, w)(g) = X(Y(g \circ f))(p)$. We want to show that this means that the function $f_{**} : T_pM \times T_pM \longrightarrow T_{f(p)}N$ is well defined and bilinear.

   First, we prove that indeed we have that $f_{**}(v, w)$ is a derivation. For this, let (x, U) be a chart on M around p and (y, V) a chart on N around $f(p)$, say N has dimension m. We have:

$$\begin{aligned}f_{**}(v,w)(g) &= X(Y(g\circ f))(p) = \sum_{i=1}^n a^i(p)\frac{\partial}{\partial x^i}\left(\sum_{j=1}^n b^j \frac{\partial(g\circ f)}{\partial x^j}\right)(p) \\ &= \sum_{i=1}^n a^i(p)\frac{\partial}{\partial x^i}\left(\sum_{j=1}^n b^j \sum_{k=1}^m \frac{\partial g}{\partial y^k}\frac{\partial f}{\partial x^j}\right)(p) \\ &= \sum_{i=1}^n a^i(p)\frac{\partial}{\partial x^i}\left(\sum_{j=1}^n \sum_{k=1}^m b^j \frac{\partial g}{\partial y^k}\frac{\partial f}{\partial x^j}\right)(p) \\ &= \sum_{i=1}^n a^i(p)\sum_{j=1}^n \sum_{k=1}^m \left(\frac{\partial}{\partial x^i}\left(b^j \frac{\partial g}{\partial y^k}\right)\frac{\partial f}{\partial x^j} + b^j \frac{\partial g}{\partial y^k}\frac{\partial^2 f}{\partial x^i \partial x^j}\right)(p) \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^m a^i(p) b^j(p)\frac{\partial^2 f(p)}{\partial x^i \partial x^j}\frac{\partial g(f(p))}{\partial y^k}\end{aligned}$$

since p being a critical point of f means that $\partial f(p)/\partial x^i = 0$ for $1 \leq i \leq n$, so the first order derivatives on f vanish. Now, since this only depends on the derivative of g at $f(p)$, we found that we can write $f_{**}(v,w)(g)$ as a sum of derivatives of g with respect to $\partial/\partial y^k$ evaluated at $f(p)$ for $1 \leq k \leq m$, which is a basis of $T_{f(p)}N$ hence $f_{**}(v,w)$ is a derivation. Moreover, if there are another vector fields $\tilde{X}$ and $\tilde{Y}$ of M such that $\tilde{X}(p) = v$ and $\tilde{Y}(p) = w$, since $g \circ f : M \longrightarrow \mathbb{R}$ we already saw in a section above that $X(Y(g \circ f))(p) = \tilde{X}(\tilde{Y}(g \circ f))(p)$. Combining this two results, we obtain that f_{**} is well defined.

Second, let $u, v, w \in T_pM$ and $r \in \mathbb{R}$, let W be a vector field on M with $W(p) = u$. Since f_{**} is well defined, we can use $W + X$ as the vector field having $W(p) + X(p) = u + v$, and then:

$$\begin{aligned} f_{**}(u+v,w)(g) &= (W+X)(Y(g \circ f))(p) = W(Y(g \circ f))(p) + X(Y(g \circ f))(p) \\ &= f_{**}(u,w)(g) + f_{**}(v,w)(g), \end{aligned}$$

analogously:

$$\begin{aligned} f_{**}(w,u+v) &= Y((W+X)(g \circ f))(p) = Y(W(g \circ f) + X(g \circ f))(p) \\ &= Y(W(g \circ f))(p) + Y(X(g \circ f))(p) \\ &= f_{**}(w,u)(g) + f_{**}(w,v)(g), \end{aligned}$$

with:

$$\begin{aligned} f_{**}(rv,w)(g) &= (rX)(Y(g \circ f))(p) = X(rY(g \circ f))(p) \\ &= rX(Y(g \circ f))(p) = rf_{**}(v,w)(g), \end{aligned}$$

analogously:

$$\begin{aligned} f_{**}(v,rw)(g) &= X((rY)(g \circ f))(p) = X(Y(r(g \circ f)))(p) \\ &= rX(Y(g \circ f))(p) = rf_{**}(v,w)(g), \end{aligned}$$

where we have used indiscriminately the definition of sum of functions and function obtained by multiplication with a scalar and that X and Y are vector fields hence evaluated at p they behave like derivations (in particular they are linear). Thus by the four equalities above, we have that f_{**} is bilinear, as desired.

5. Let $c : \mathbb{R} \longrightarrow M$ have 0 as a critical point. Consider $(t, \mathbb{R})$ a chart of $\mathbb{R}$, this yields $\partial/\partial t$ a basis in $T_0\mathbb{R}$, meaning that $(1,1) \in T_0\mathbb{R}$ can be written as $(\partial/\partial t, \partial/\partial t)$, meaning that using the sections above we have for any $f : M \longrightarrow \mathbb{R}$ smooth:

$$\begin{aligned} c_{**}(1,1)(f) &= c_{**}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)(f) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}(f \circ c)\right)(0) \\ &= (f \circ c)''(0) = c''(0)(f) \end{aligned}$$

and $c_{**}(1,1) = c''(0)$ as desired.

Exercise 2

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ smooth with $f(0) = 0$, define $g(t) = f(\sqrt{t})$ for $t \geq 0$. First, we notice that by Taylor's Theorem with the Peano form of the reminder we can write:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \alpha(x)x^3 = f(0) + \frac{f''(0)}{2}x^2 + \alpha(x)x^3$$

with $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ a smooth function with $\lim_{x\to 0} \alpha(x) = 0$. Now, we have that:

$$\begin{aligned} g'_+(0) &= \lim_{h\downarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h\downarrow 0} \frac{f(\sqrt{h}) - f(0)}{h} \\ &= \lim_{h\downarrow 0} \frac{f(0) + \frac{f''(0)}{2}h + \alpha(\sqrt{h})\sqrt{h^3} - f(0)}{h} = \frac{f''(0)}{2} + \lim_{h\downarrow 0} \frac{\alpha(\sqrt{h})\sqrt{h^3}}{h} \\ &= \frac{f''(0)}{2} + \lim_{h\downarrow 0} \alpha(\sqrt{h})\sqrt{h} = \frac{f''(0)}{2} + \lim_{\sqrt{h}\downarrow 0} \alpha(\sqrt{h})\sqrt{h} = \frac{f''(0)}{2}, \end{aligned}$$

where we have used that $h \downarrow 0$ if and only if $\sqrt{h} \downarrow 0$ and both $\lim_{x\downarrow 0} \alpha(x) = \lim_{x\downarrow 0} x = 0$. This proves the desired result.

2. Given $c : \mathbb{R} \longrightarrow M$ smooth with $c'(0) = 0 \in T_pM$ we define $\gamma(t) = c(\sqrt{t})$ for $t \geq 0$. For any fixed function $f : M \longrightarrow \mathbb{R}$ smooth, consider now $f \circ c : \mathbb{R} \longrightarrow \mathbb{R}$, as in the section above we set $g(t) = (f \circ c)(\sqrt{t})$ for $t \geq 0$. Now:

$$c''(0)(f) = (f \circ c)''(0) = 2g'_+(0) = 2(f \circ \gamma)'(0) = 2\gamma'(0)(f),$$

where we have used the definition of $c''(0)$ in the first equality, the result proven in the section above in the second equality, notice how $g(t) = (f \circ \gamma)(t)$ for the third equality and the definition of $\gamma'(0)$ in the fourth equality. Since this is true for every f, we have that $c''(0) = 2\gamma'(0)$, as desired.

Exercise 3

1. Consider the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $f(x, y, z) = f(y, -x, z)$. Let $p = (0, 0, a)$ for $a \in \mathbb{R}$, we compute $f_* : T_p\mathbb{R}^3 \longrightarrow \mathbb{R}^3$. Since this is the differential of f evaluated at p, we have:

$$f_* = (Df)_p = \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}_p = \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$

   for every $p = (0, 0, a)$ with $a \in \mathbb{R}$. Moreover, we can think of f_* as belonging to $\mathbb{R}^3 \otimes (\mathbb{R}^3)^*$: if we consider e_1, e_2, e_3 the canonical basis of $\mathbb{R}^3$, with e_1^*, e_2^*, e_3^* the dual basis of $(\mathbb{R}^3)^*$, then $e_i \otimes e_j^*$ is the multiplication of a column vector with a row vector, yielding a matrix with a one in the entry (i, j) and zeros everywhere else, meaning that summing over the basis $\{e_i \otimes e_j^*\}_{i,j=1,2,3}$ of $\mathbb{R}^3 \otimes (\mathbb{R}^3)^*$ weighted by the entries of f_* we obtain:

$$\begin{aligned} (1) \cdot (e_1 \otimes e_2^*) + (-1) \cdot (e_2 \otimes e_1^*) + (1) \cdot (e_3 \otimes e_3^*) \quad &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = f_* \end{aligned}$$

   and we gave f_* in the basis of $\mathbb{R}^3 \otimes (\mathbb{R}^3)^*$ with the desired components.

2. We consider now f_* evaluated at $(0, 0, \pm 1)$. We already saw that $(Df)_{(0,0,a)}$ does not depend on $a \in \mathbb{R}$, hence we obtain the same f_* as above, and its expression as a $(1, 1)$-tensor is the above given.

3. Let $f : M \longrightarrow N$ is a smooth function between manifolds M, N of dimensions m and n respectively. Let $p \in M$, we want to describe $f_* : T_pM \longrightarrow T_{f(p)}N$ as an element of $T_{f(p)}N \otimes T_p^*M$ and give its components. In order to do this, we will notice that using local charts, say (x, U) near $p \in M$ and (y, V) near $f(p) \in N$, we can assume that $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. In particular, T_pM has $\partial/\partial x^1, \ldots, \partial/\partial x^m$ as a basis and $T_{f(p)}N$ has $\partial/\partial y^1, \ldots, \partial/\partial y^n$ a basis. Thus $(f_*)_p = (Df)_p$ is a matrix eating an m-dimensional vector in T_pM and pooping an n-dimensional vector in $T_{f(p)}N$, that is, it has m columns and n rows, say $(f_*)_p = (a_{i,j}^p)_{i,j}$ with $i = 1, \ldots, n$ and $j = 1, \ldots, m$ its expression with the entries in the basis above. Now, T_p^*M has $dx^1, \ldots, dx^m$ as the canonical dual basis. In particular, as in the section above, $\{\partial/\partial y^i \otimes dx^j\}_{i,j}$ with $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is the corresponding basis of $T_{f(p)}N \otimes T_p^*M$ and $\partial/\partial y^i \otimes dx^j$ is the multiplication of a column vector with a row vector, yielding a matrix with a one in the entry (i, j) and zeros everywhere else. Thus we can again rewrite:

$$(f_*)_p = \sum_{i,j} a_{i,j}^p \cdot \left(\frac{\partial}{\partial y^i} \otimes dx^j \right)$$

writing $f_* \in T_{f(p)}N \otimes T_p^*M$ with the entries of f_* as its components, what we desired.

Exercise 4

1. Let $\phi : M \longrightarrow N$ and $f : N \longrightarrow \mathbb{R}$ be smooth, Y a vector field on M. Now:

$$\phi^*(df)(Y) = d(f \circ \phi)(Y) = Y(f \circ \phi),$$

where we used [2, Lemma 6.12 (p. 137)] and the definition of the action on a vector field.

2. Using the limit definition, we have that:

$$\begin{aligned}
(L_X(df)(p))(Y_p) &= \left(\lim_{h\to 0} \frac{\phi_h^*(df)(p) - (df)(p)}{h}\right)(Y_p) \\
&= \lim_{h\to 0} \frac{\phi_h^*(df)(p)(Y_p) - (df)(p)(Y_p)}{h} \\
&= \lim_{h\to 0} \frac{Y(f \circ \phi_h)(p) - Y(f)(p)}{h} \\
&= Y\left(\lim_{h\to 0} \frac{(f \circ \phi_h)(p) - f(p)}{h}\right) \\
&= Y\left(\lim_{h\to 0} \frac{(\phi_h^* f)(p) - f(p)}{h}\right) \\
&= Y(X(f))(p) = Y_p(L_X f)
\end{aligned}$$

meaning that:

$$d(L_X f)(Y_p) = Y_p(L_X f) = (L_X(df)(p))(Y_p)$$

and thus $d(L_X f) = L_X(df)$.

3. Let X and Y vector fields on M and $f : M \longrightarrow \mathbb{R}$ a smooth function. Let X generate the family $\{\phi_t\}$, set $\alpha(t,h) = Y_{\phi_{-t}(p)}(f \circ \phi_h)$. Now:

$$\begin{aligned}
D_1\alpha(0,0) &= \lim_{t\to 0} \frac{\alpha(t,0) - \alpha(0,0)}{t} = \lim_{t\to 0} \frac{Y_{\phi_{-t}(p)}(f) - Y_p(f)}{t} \\
&= \lim_{t\to 0} \frac{Y(f)(\phi_{-t}(p)) - Y(f)(p)}{t} = -\lim_{t\to 0} \frac{Y(f)(\phi_t(p)) - Y(f)(p)}{t} \\
&= -X(Y(f))(p) = -X_p(Y(f)),
\end{aligned}$$

where the only trick we used is noting that since $t \to 0$, the change $t \mapsto -t$ yields the fourth equality. Moreover:

$$\begin{aligned}
D_2\alpha(0,0) &= \lim_{h\to 0} \frac{\alpha(0,h) - \alpha(0,0)}{h} = \lim_{h\to 0} \frac{Y_p(f \circ \phi_h) - Y_p(f)}{h} \\
&= Y_p\left(\lim_{h\to 0} \frac{f \circ \phi_h - f}{h}\right) = Y_p(X(f)).
\end{aligned}$$

Thus if we have $c(h) = \alpha(h,h)$ then:

$$\begin{aligned} -c'(0) &= -\lim_{h\to 0}\frac{\alpha(h,h)-\alpha(0,0)}{h} = -\lim_{h\to 0}\frac{\alpha(h,h)-\alpha(h,0)}{h} \\ &- \lim_{h\to 0}\frac{\alpha(h,0)-\alpha(0,0)}{h} = -D_2\alpha(0,0) - D_1\alpha(0,0) = -Y_p(X(f)) \\ &+ X_p(Y(f)) = [X,Y]_p(f) = L_X Y(p)(f)), \end{aligned}$$

the desired results.

Exercise 5

Let $\gamma : J \longrightarrow M$ be an integral curve of a vector field X on a manifold M with $\gamma'(t_0) = 0$ for some $t_0 \subset J \subset \mathbb{R}$. We show that $\gamma(t) = \gamma(t_0)$ for every $t \in J$.

By definition of γ being an integral curve, it means that it is a solution of the system:

$$\begin{cases} \frac{d\rho(s)}{dt} = X(\rho(s)) \forall s \in J \\ \rho(t_0) = \gamma(t_0) \end{cases} .$$

However, the curve $\beta : J \longrightarrow M$ defined by $\beta(t) = \gamma(t_0)$ is also a solution of the system above since:

$$\begin{cases} \frac{d\beta(s)}{dt} = 0 = \frac{d\alpha(t_0)}{dt} X(\alpha(t_0)) = X(\beta(s)) \forall s \in J \\ \rho(t_0) = \gamma(t_0) \end{cases} ,$$

where we have used that β is constant thus has zero derivative. Now we have α and β two solutions of the same system, by [1, Theorem 2 (p. 141)] we have uniqueness of the solutions of ordinary differential equations. This means that in fact $\alpha = \beta$ as integral curves, thus $\alpha(t) = \beta(t) = \alpha(t_0)$ for every $t \in J$, what we wanted to prove.

References

[1] M. Spivak, *A Comprehensive Introduction to Differential Geometry - Volume 1*, Publish or Perish INC., 2005.

[2] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2003.