Differential Geometry I - Homework 6

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Exercise 1

1. We consider $\mathbb{R}^2$ with the standard coordinates. Let $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, we want to find an expression for R in polar coordinates. We know that such coordinates are:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

meaning that for a general smooth function $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ we have:

$$\frac{\partial u(x,y)}{\partial r} = \frac{\partial u(x,y)}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u(x,y)}{\partial x}\cos(\theta) + \frac{\partial u(x,y)}{\partial y}\sin(\theta)$$

hence by multiplying everything by r, we obtain:

$$r\frac{\partial u(x,y)}{\partial r} = r\cos(\theta)\frac{\partial u(x,y)}{\partial x} + r\sin(\theta)\frac{\partial u(x,y)}{\partial y} = x\frac{\partial u(x,y)}{\partial x} + y\frac{\partial u(x,y)}{\partial y}$$

meaning that $R = r\frac{\partial}{\partial r}$ in polar coordinates.

2. We consider the fields $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y} + f\frac{\partial}{\partial z}$ with $f(x,y,z) = xz + \frac{x^3}{3} + xy^2 + x^3y^2$.

   (a) We have:

$$\begin{aligned}
[X,Y] &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} + f\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) + f\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\right) \\
&= \frac{\partial^2}{\partial x\partial z} + \frac{\partial f}{\partial x}\frac{\partial}{\partial z} + f\frac{\partial^2}{\partial x\partial z} - \frac{\partial^2}{\partial y\partial x} + f\frac{\partial^2}{\partial z\partial x} = \frac{\partial f}{\partial x}\frac{\partial}{\partial z} \\
&= (z + x^2 + y^2 + 3x^2y^2)\frac{\partial}{\partial z}
\end{aligned}$$

   we rename $g(x,y,z) = z + x^2 + y^2 + 3x^2y^2$. Now:

$$\begin{aligned}
[X,[X,Y]] &= \frac{\partial}{\partial x}\left(g\frac{\partial}{\partial z}\right) - g\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}\right) \\
&= \frac{\partial g}{\partial x}\frac{\partial}{\partial z} + g\frac{\partial^2}{\partial x\partial z} - g\frac{\partial^2}{\partial z\partial x} = \frac{\partial g}{\partial y}\frac{\partial}{\partial z} = (2x + 6xy^2)\frac{\partial}{\partial z}
\end{aligned}$$

   and:

$$\begin{aligned}
[Y,[X,Y]] &= \frac{\partial}{\partial y}\left(g\frac{\partial}{\partial z}\right) + f\frac{\partial}{\partial z}\left(g\frac{\partial}{\partial z}\right) - g\frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\right) - g\frac{\partial}{\partial z}\left(f\frac{\partial}{\partial z}\right) \\
&= \frac{\partial g}{\partial y}\frac{\partial}{\partial z} + g\frac{\partial^2}{\partial y\partial z} + f\frac{\partial g}{\partial z}\frac{\partial}{\partial z} + fg\frac{\partial^2}{\partial z^2} - g\frac{\partial}{\partial z\partial y} - g\frac{\partial f}{\partial z}\frac{\partial}{\partial z} \\
&- gf\frac{\partial^2}{\partial z^2} = \frac{\partial g}{\partial y}\frac{\partial}{\partial z} + f\frac{\partial g}{\partial z}\frac{\partial}{\partial z} - g\frac{\partial f}{\partial z}\frac{\partial}{\partial z} \\
&= (2y + 6x^2y - \frac{2x^3}{3} - 2x^3y^3)\frac{\partial}{\partial z}.
\end{aligned}$$

(b) We now find the points $p \in \mathbb{R}^3$ such that $[X,Y]_p \in \langle X_p, Y_p \rangle$. We will take $p = (x,y,z)$, without risk of confusing notation since the differentials will never be applied to functions. Thus we want to find $a, b \in \mathbb{R}$ such that $[X,Y]_p = aX_p + bY_p$. Notice how X_p has a term $\frac{\partial}{\partial x}$ that does not appear in $[X,Y]_p$ nor Y_p, and since this is an element of the canonical basis, we must have $a = 0$. Similarly, the term $\frac{\partial}{\partial y}$ of the canonical basis cannot be matched in $[X,Y]_p$, thus we must have $b = 0$. Since $\frac{\partial}{\partial z}$ is an element of the canonical basis (hence non zero), we must have $z + x^2 + y^2 + 3x^2y^2 = 0$. Thus the desired points are the ones of the form $p = (x, y, -x^2 - y^2 - 3x^2y^2)$ with $x, y \in \mathbb{R}$.

(c) We now find the points $p \in \mathbb{R}^3$ such that both $[X,[X,Y]]_p, [Y,[X,Y]]_p \in \langle X_p, Y_p \rangle$. We will again take $p = (x,y,z)$ and an analogous argument as the one above finds that the coefficients of both X_p and Y_p in the expressions of both $[X,[X,Y]]_p$ and $[Y,[X,Y]]_p$ must be zero. Hence the two remaining expressions are:

$$\begin{cases} 2x + 6xy^2 = 0 \\ 2y + 6x^2y - \frac{2x^3}{3} - 2x^3y^2 = 0 \end{cases}$$

and the first equation yields $2x(1+3y^2) = 0$, but since $y \in \mathbb{R}$ we always have $1 + 3y^2 \neq 0$, meaning that we must have $x = 0$. Imposing this to the second equation immediately yields $2y = 0$ hence $y = 0$. Notice how we have no restrictions for the third component of the points. Thus the desired points are the ones of the form $p = (0,0,z)$ with $z \in \mathbb{R}$.

Exercise 2

Let Δ be the distribution on $\mathbb{R}^3 \setminus \{0\}$ given at a point $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$ by $\Delta_{(x,y,z)} = \{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} : ax + by + cz = 0\}$. To show that Δ is integrable, we will show that given $X, Y \in \Delta$, we have $[X, Y] \in \Delta$. Assuming we already proved this, we are under the hypothesis of the Frobenius Integrability Theorem [2, Theorem 19.10 (p. 501)], and applying it we directly obtain that Δ is integrable, as desired.

Hence the only thing we have to do is check that if $X, Y \in \Delta$ then $[X, Y] \in \Delta$. The general expression of such vector fields belonging to Δ is:

$$X = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z}, \quad Y = g_1\frac{\partial}{\partial x} + g_2\frac{\partial}{\partial y} + g_3\frac{\partial}{\partial z}$$

where $f_i, g_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$ are smooth functions for $i = 1, 2, 3$, such that for every $p = (p_1, p_2, p_3) \in \mathbb{R}^3 \setminus \{0\}$ we have:

$$f_1(p)p_1 + f_2(p)p_2 + f_3(p)p_3 = 0, \quad g_1(p)p_1 + g_2(p)p_2 + g_3(p)p_3 = 0$$

and in particular the equations:

$$f_1(x, y, z)x + f_2(x, y, z)y + f_3(x, y, z)z = 0, \; g_1(x, y, z)x + g_2(x, y, z)y + g_3(x, y, z)z = 0$$

are true as functions from $\mathbb{R}^3 \setminus \{0\}$ to $\mathbb{R}$. This will be particularly relevant since we will need to differentiate them. Now, we proceed to compute $[X, Y]_p$, and for this we first note that it must be a derivation since X and Y being vector fields means that $[X, Y]$ is a vector field, which evaluated at a point yields a derivation. This means that we can just compute the first order derivatives and ignore the rest of higher order derivatives. Computing (we will then subtract $X(Y) - Y(X)$ of these relevant terms):

$$\begin{aligned}
X(Y) \text{ yields} \qquad & f_1\left(\frac{\partial g_1}{\partial x}\frac{\partial}{\partial x} + \frac{\partial g_2}{\partial x}\frac{\partial}{\partial y} + \frac{\partial g_3}{\partial x}\frac{\partial}{\partial z}\right) \\
+ \; & f_2\left(\frac{\partial g_1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g_2}{\partial y}\frac{\partial}{\partial y} + \frac{\partial g_3}{\partial y}\frac{\partial}{\partial z}\right) \\
+ \; & f_3\left(\frac{\partial g_1}{\partial z}\frac{\partial}{\partial x} + \frac{\partial g_2}{\partial z}\frac{\partial}{\partial y} + \frac{\partial g_3}{\partial z}\frac{\partial}{\partial z}\right) \\
Y(X) \text{ yields} \qquad & g_1\left(\frac{\partial f_1}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial x}\frac{\partial}{\partial y} + \frac{\partial f_3}{\partial x}\frac{\partial}{\partial z}\right) \\
+ \; & g_2\left(\frac{\partial f_1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial y}\frac{\partial}{\partial y} + \frac{\partial f_3}{\partial y}\frac{\partial}{\partial z}\right) \\
+ \; & g_3\left(\frac{\partial f_1}{\partial z}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial z}\frac{\partial}{\partial y} + \frac{\partial f_3}{\partial z}\frac{\partial}{\partial z}\right)
\end{aligned}$$

meaning that the condition of $[X,Y]_p \in \Delta_p$ is, once we evaluate at a point $p=(p_1,p_2,p_3)$:

$$\begin{aligned}
& \left(f_1(p)\frac{\partial g_1}{\partial x}|_p + f_2(p)\frac{\partial g_1}{\partial y}|_p + f_3(p)\frac{\partial g_1}{\partial z}|_p - g_1(p)\frac{\partial f_1}{\partial x}|_p - g_2(p)\frac{\partial f_1}{\partial y}|_p - g_3(p)\frac{\partial f_1}{\partial z}|_p \right) p_1 \\
+ & \left(f_1(p)\frac{\partial g_2}{\partial x}|_p + f_2(p)\frac{\partial g_2}{\partial y}|_p + f_3(p)\frac{\partial g_2}{\partial z}|_p - g_1(p)\frac{\partial f_2}{\partial x}|_p - g_2(p)\frac{\partial f_2}{\partial y}|_p - g_3(p)\frac{\partial f_2}{\partial z}|_p \right) p_2 \\
+ & \left(f_1(p)\frac{\partial g_3}{\partial x}|_p + f_2(p)\frac{\partial g_3}{\partial y}|_p + f_3(p)\frac{\partial g_3}{\partial z}|_p - g_1(p)\frac{\partial f_3}{\partial x}|_p - g_2(p)\frac{\partial f_3}{\partial y}|_p - g_3(p)\frac{\partial f_3}{\partial z}|_p \right) p_3
\end{aligned}$$

and we want this to be zero. To obtain so, we come back to the two equations we said we would need to differentiate, and proceed to do so with respect to every coordinate. This results in (as above, we have also evaluated at the pertinent point $p=(p_1,p_2,p_3)$):

$$\begin{array}{rcll}
\frac{\partial f_1}{\partial x}|_p p_1 + f_1(p) + \frac{\partial f_2}{\partial x}|_p p_2 + \frac{\partial f_3}{\partial x}|_p p_3 & = & 0 & \cdot -g_1(p) \\
\frac{\partial f_1}{\partial y}|_p p_1 + \frac{\partial f_2}{\partial y}|_p p_2 + f_2(p) + \frac{\partial f_3}{\partial y}|_p p_3 & = & 0 & \cdot -g_2(p) \\
\frac{\partial f_1}{\partial z}|_p p_1 + \frac{\partial f_2}{\partial z}|_p p_2 + \frac{\partial f_3}{\partial z}|_p p_3 + f_3(p) & = & 0 & \cdot -g_3(p) \\
\frac{\partial g_1}{\partial x}|_p p_1 + g_1(p) + \frac{\partial g_2}{\partial x}|_p p_2 + \frac{\partial g_3}{\partial x}|_p p_3 & = & 0 & \cdot f_1(p) \\
\frac{\partial g_1}{\partial y}|_p p_1 + \frac{\partial g_2}{\partial y}|_p p_2 + g_2(p) + \frac{\partial g_3}{\partial y}|_p p_3 & = & 0 & \cdot f_2(p) \\
\frac{\partial g_1}{\partial z}|_p p_1 + \frac{\partial g_2}{\partial z}|_p p_2 + \frac{\partial g_3}{\partial z}|_p p_3 + g_3(p) & = & 0 & \cdot f_3(p)
\end{array}$$

where we have already written in a right column the term with which we intend to multiply each equation. We now notice that if we multiply as stated and then sum all six equations, on the left hand side we precisely obtain the expression above (that arose from evaluating $[X,Y]_p$ and trying to check if it belonged in Δ_p) since the multiplications $f_i(p)g_j(p)$ for $i,j=1,2,3$ appear twice with opposite signs, hence they cancel out. Since the right hand side is always zero, this means that:

$$\begin{aligned}
& \left(f_1(p)\frac{\partial g_1}{\partial x}|_p + f_2(p)\frac{\partial g_1}{\partial y}|_p + f_3(p)\frac{\partial g_1}{\partial z}|_p - g_1(p)\frac{\partial f_1}{\partial x}|_p - g_2(p)\frac{\partial f_1}{\partial y}|_p - g_3(p)\frac{\partial f_1}{\partial z}|_p \right) p_1 \\
+ & \left(f_1(p)\frac{\partial g_2}{\partial x}|_p + f_2(p)\frac{\partial g_2}{\partial y}|_p + f_3(p)\frac{\partial g_2}{\partial z}|_p - g_1(p)\frac{\partial f_2}{\partial x}|_p - g_2(p)\frac{\partial f_2}{\partial y}|_p - g_3(p)\frac{\partial f_2}{\partial z}|_p \right) p_2 \\
+ & \left(f_1(p)\frac{\partial g_3}{\partial x}|_p + f_2(p)\frac{\partial g_3}{\partial y}|_p + f_3(p)\frac{\partial g_3}{\partial z}|_p - g_1(p)\frac{\partial f_3}{\partial x}|_p - g_2(p)\frac{\partial f_3}{\partial y}|_p - g_3(p)\frac{\partial f_3}{\partial z}|_p \right) p_3 \\
= & \ 0
\end{aligned}$$

and thus $[X,Y]_p \in \Delta_p$ for every $p \in \mathbb{R}^3 \setminus \{0\}$, meaning that $[X,Y] \in \Delta$, as we wanted to prove.

Exercise 3

This exercise is concerned with finding a solution $\alpha : W \longrightarrow V$ for the differential equation:

$$\begin{cases} \alpha(0) & = x \\ \frac{\partial \alpha}{\partial t^j}(t) & = f_j(t, \alpha(t)) \end{cases}$$

for all $t \in W \subset \mathbb{R}^m$.

1. We want to have $\alpha(ut) = \beta(u,t)$ for some function $\beta : [0,\epsilon) \times W \longrightarrow V$. Now, this β must cleary satisfy $\beta(0,t) = \alpha(0t) = \alpha(0) = x$ as initial condition and:

$$\frac{\partial \beta}{\partial u}(u,t) = \frac{\partial \alpha}{\partial u}(ut) = \sum_{j=1}^{m} \frac{\partial \alpha}{\partial t^j}(ut) \frac{\partial (ut^j)}{\partial u}(t) = \sum_{j=1}^{m} f_j(ut, \alpha(ut))t^j = \sum_{j=1}^{m} t^j f_j(ut, \beta(u,t))$$

   where we used the chain rule for differentiation and the differential equation that we want α to satisfy. This is the desired result. As a side note, we know that such a differential equation can be solved, and thus such β exists (although there may be different ϵ's involved for the domain).

2. Now we will show that we can use a fixed value for ϵ. We will prove that $\beta(u, vt)$ and $\beta(uv, t)$ satisfy the same differential equation as functions of u. First, we clearly have $\beta(0, vt) = x$ and $\beta(0v, t) = \beta(0, t) = x$ as initial condition. We first compute:

$$\frac{\partial \beta(u,vt)}{\partial u} = \sum_{j=1}^{m} (vt)^j f_j(uvt, \beta(u,vt)) = v \sum_{j=1}^{m} t^j f_j(uvt, \beta(u,vt))$$

   and then:

$$\frac{\partial \beta(uv,t)}{\partial u} = \frac{\partial \beta(uv,t)}{\partial (uv)} \frac{\partial (uv)}{\partial u} = v \sum_{j=1}^{m} t^j f_j(uvt, \beta(uv,t))$$

   where in both cases we have used the expression obtained in the section above. This means that both functions satisfy the differential equation:

$$\begin{cases} \psi(0) & = x \\ \frac{\partial \psi}{\partial u}(u) & = v \sum_{j=1}^{m} t^j f_j(uvt, \psi(u)) \end{cases}$$

   hence by uniqueness of the solution, we must have $\beta(u, vt) = \beta(uv, t)$, as desired.

3. Using the equality found in the section above, we differentiate with respect to t^j each side:

$$\begin{aligned} v \frac{\partial \beta}{\partial t^j}(u,vt) = \sum_{i=1}^{m} \frac{\partial \beta}{\partial t^i}(u,vt) \frac{\partial (vt^i)}{\partial t^j} \quad &= \quad \frac{\partial \beta(u,vt)}{\partial t^j} \\ &= \quad \frac{\partial \beta(uv,t)}{\partial t^j} = \sum_{i=1}^{m} \frac{\partial \beta}{\partial t^i}(uv,t) \frac{\partial t^i}{\partial t^j} = \frac{\partial \beta}{\partial t^j}(uv,t) \end{aligned}$$

hence by substituting $u = 1$ we obtain $v\frac{\partial\beta}{\partial t^j}(1, vt) = \frac{\partial\beta}{\partial t^j}(v, t)$, as desired.

4. We prove, using the integrability condition on f, that the functions $\frac{\partial\beta}{\partial t^j}(v, t)$ and $vf_j(vt, \beta(v, t))$ satisfy the same differential equation as functions of v. By uniqueness of the solutions, we will then have that those two expressions are equal. First, using the section above we have that $\frac{\partial\beta}{\partial t^j}(0, t) = 0\frac{\partial\beta}{\partial t^j}(1, 0t) = 0$, and $0f_j(0t, \beta(0, t)) = 0$ as initial condition. Second, we will just differentiate each of the functions with respect to v. We have:

$$\begin{aligned}
\frac{\partial}{\partial v}\left(\frac{\partial\beta}{\partial t^j}(v,t)\right)(v) &= \frac{\partial}{\partial t^j}\left(\frac{\partial\beta}{\partial v}(v,t)\right)(v) = \frac{\partial}{\partial t^j}\left(\sum_{i=1}^m t^i f_i(vt,\beta(v,t))\right)(v) \\
&= \sum_{i=1}^m t^i \frac{\partial f_i(vt,\beta(v,t))}{\partial t^j}(v) + f_j(vt,\beta(v,t)) = f_j(vt,\beta(v,t)) \\
&+ \sum_{i=1}^m t^i \left(v\frac{\partial f_i}{\partial t^j}(vt,\beta(v,t)) + \sum_{k=1}^n \frac{\partial f_i}{\partial x^k}(vt,\beta(v,t))\frac{\partial\beta^k}{\partial t^j}(v,t)\right)
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\partial(vf_j(vt,\beta(v,t)))}{\partial v}(v) &= f_j(vt,\beta(v,t)) + v\frac{\partial f_j(vt,\beta(v,t))}{\partial v}(v) \\
&= f_j(vt,\beta(v,t)) + v\left(\sum_{i=1}^m \frac{\partial f_j}{\partial t^i}(vt,\beta(v,t))t^i\right. \\
&+ \left.\sum_{k=1}^n \frac{\partial f_j}{\partial x^k}(vt,\beta(v,t))\frac{\partial\beta^k}{\partial v}(v,t)\right) = f_j(vt,\beta(v,t)) \\
&+ v\left(\sum_{i=1}^m \frac{\partial f_j}{\partial t^i}(vt,\beta(v,t))t^i\right. \\
&+ \left.\sum_{k=1}^n \frac{\partial f_j}{\partial x^k}(vt,\beta(v,t))\left(\sum_{i=1}^m t^i f_i^k(vt,\beta(v,t))\right)\right) \\
&= f_j(vt,\beta(v,t)) + \sum_{i=1}^m t^i v\left(\frac{\partial f_j}{\partial t^i}(vt,\beta(v,t))\right. \\
&+ \left.\sum_{k=1}^n \frac{\partial f_j}{\partial x^k}(vt,\beta(v,t))f_i^k(vt,\beta(v,t))\right)
\end{aligned}$$

where we have abused all the tricks of differentiation known to mankind (specially the notation of the evaluation of the differentials on the point) and used the results proven in the sections above. We note that the sum we obtained in the second differentiation does not exactly resemble the one obtained in the first, since it appears that the indexes i and j are in the wrong order. However, the integrability condition on f gives us the equality:

$$\frac{\partial f_j}{\partial t^i} + \sum_{k=1}^{n} \frac{\partial f_j}{\partial x^k} f_i^k = \frac{\partial f_i}{\partial t^j} + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x^k} f_j^k$$

and substituting this we obtain:

$$\begin{aligned}
\frac{\partial(v f_j(vt, \beta(v,t)))}{\partial v}(v,t) &= f_j(vt, \beta(v,t)) + \sum_{i=1}^{m} t^i v \left(\frac{\partial f_i}{\partial t^j}(vt, \beta(v,t)) \right. \\
&+ \left. \sum_{k=1}^{n} \frac{\partial f_i}{\partial x^k}(vt, \beta(v,t)) f_j^k(vt, \beta(v,t)) \right) \\
&= f_j(vt, \beta(v,t)) + \sum_{i=1}^{m} t^i \left(v \frac{\partial f_i}{\partial t^j}(vt, \beta(v,t)) \right. \\
&+ \left. \sum_{k=1}^{n} \frac{\partial f_i}{\partial x^k}(vt, \beta(v,t)) (v f_j(vt, \beta(v,t)))^k \right).
\end{aligned}$$

This means that both functions satisfy the differential equation:

$$\begin{cases} \psi(0) = 0 \\ \frac{\partial \psi}{\partial v}(v) = f_j(vt, \beta(v,t)) + \sum_{i=1}^{m} t^i \left(v \frac{\partial f_i}{\partial t^j}(vt, \beta(v,t)) + \sum_{k=1}^{n} \frac{\partial f_i}{\partial x^k}(vt, \beta(v,t)) \psi^k(v) \right) \end{cases}$$

where ψ^k denotes the k-th component of ψ, and we obtained the desired result.

5. We finally define $\alpha(t) = \beta(1,t)$. We have $\alpha(vt) = \beta(1, vt) = \beta(v,t)$ using one of the sections above. Moreover $\alpha(0) = \beta(0t) = \beta(0,t) = x$ and:

$$\frac{\partial \alpha}{\partial t^j}(t) = \frac{\partial \beta}{\partial t^j}(1,t) = 1 f_j(1t, \beta(1,t)) = f_j(t, \alpha(t)),$$

where we have used the definition of α and the proved above. Hence α satisfies the differential equation:

$$\begin{cases} \alpha(0) &= x \\ \frac{\partial \alpha}{\partial t^j}(t) &= f_j(t, \alpha(t)) \end{cases}$$

for all $t \in W \subset \mathbb{R}^m$, as desired.

Exercise 4

An element $w \in \Lambda^k(V)$ is called decomposable if we can write $w = \phi_1 \wedge \cdots \wedge \phi_k$ for some $\phi, \dots, \phi_k \in \Lambda^1(V)$.

1. If $\dim(V) \leq 3$, then every $w \in \Lambda^2(V)$ is decomposable. This is clear if $\dim(V) = 0, 1$ since $2 > 1, 0$ and thus in those cases $\Lambda^2(V)$ is trivial. If $\dim(V) = 2$, say x, y are a basis, then $\Lambda^2(V) = \langle dx \wedge dy \rangle$ with $dx, dy \in \Lambda^1(V)$, hence any element in $\Lambda^2(V)$ is clearly decomposable (we may have to use the multilinearity of the wedge product). Finally for $\dim(V) = 3$, say x, y, z are a basis, then $\Lambda^2(V) = \langle dx \wedge dy, dy \wedge dz, dz \wedge dx \rangle$ thus a general element $w \in \Lambda^2(V)$ has the form $w = w_1 dx \wedge dy + w_2 dy \wedge dz + w_3 dz \wedge dx$ for certain $w_1, w_2, w_3 \in \mathbb{R}$. Since $\Lambda^1(V) = \langle dx, dy, dz \rangle$, this means that the expression of two general elements $\phi_1, \phi_2 \in \Lambda^1(V)$ is of the form $\phi_a = a_1 dx + a_2 dy + a_3 dz$ and $\phi_b = b_1 dx + b_2 dy + b_3 dz$ with $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$. Now, if we compute the wedge and impose $\phi_a \wedge \phi_b = w$, we obtain:

$$\begin{aligned}(a_1 b_2 - b_1 a_2) dx \wedge dy + (a_2 b_3 - b_2 a_3) dy \wedge dz + (a_3 b_1 - b_3 a_1) dz \wedge dx &= \phi_a \wedge \phi_b \\ &= w = w_1 dx \wedge dy + w_2 dy \wedge dz + w_3 dz \wedge dx\end{aligned}$$

that is the linear system of equations over the real numbers:

$$\begin{cases} a_1 b_2 - b_1 a_2 = w_1 \\ a_2 b_3 - b_2 a_3 = w_2 \\ a_3 b_1 - b_3 a_1 = w_3 \end{cases}$$

which has three equations and six variables, meaning that not only it has a solution, but it has an infinite number of solutions: we have extra degrees of freedom. Thus there exist $\phi_a, \phi_b \in \Lambda^1(V)$ having as components the solutions of the system above, and by construction we now have $w = \phi_a \wedge \phi_b$, that is, w is decomposable.

2. If we have $\phi_1, \phi_2, \phi_3, \phi_4$ linearly independent, then $w = (\phi_1 \wedge \phi_2) + (\phi_3 \wedge \phi_4)$ is not decomposable. To see this, we compute:

$$\begin{aligned} w \wedge w &= (\phi_1 \wedge \phi_2) \wedge (\phi_1 \wedge \phi_2) + (\phi_1 \wedge \phi_2) \wedge (\phi_3 \wedge \phi_4) \\ &+ (\phi_3 \wedge \phi_4) \wedge (\phi_1 \wedge \phi_2) + (\phi_3 \wedge \phi_4) \wedge (\phi_3 \wedge \phi_4) \\ &= 2 \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 \neq 0 \end{aligned}$$

where for the very last equality we have used [1, Corollary 4 (p. 206)], recalling that the terms of the wedge are linearly independent. However, suppose w is decomposable, that is, there are $\eta, \nu \in \Lambda^1(V)$ such that $w = \eta \wedge \nu$. Then $w \wedge w = (\eta \wedge \nu) \wedge (\eta \wedge \nu) = 0$. Thus w being decomposable yields a contradiction, meaning that w is not decomposable, as desired.

Exercise 5

Let $\phi_1, \ldots, \phi_k \in \Lambda^1(V)$ be linearly independent, let $\psi_1, \ldots, \psi_k \in \Lambda^1(V)$ satisfy $(\phi_1 \wedge \psi_1) + \cdots + (\phi_k \wedge \psi_k) = 0$. First, choose a fixed $j = 1, \ldots, k$. We have that wedging the above equality with $\bigwedge_{i=1, i\neq j}^k \phi_i$ we obtain:

$$0 = \bigwedge_{i=1, i\neq j}^{k} \phi_i \wedge \left(\sum_{i=1}^{k} \phi_i \wedge \psi_i \right) = \bigwedge_{i=1, i\neq j}^{k} \phi_i \wedge (\phi_j \wedge \psi_j) = \pm \phi_1 \wedge \cdots \wedge \phi_k \wedge \psi_j$$

where we have used that the wedge product is zero when we have repeating factors and that to permuting two factors we have to multiply by -1, meaning that to put ϕ_j in its position we may need to add a ± 1 factor to the equality. However, since on the left hand side we have a zero, this factor doesn't matter and for every $j = 1, \ldots, k$ we obtain $0 = \phi_1 \wedge \cdots \wedge \phi_k \wedge \psi_j$. In virtue of [1, Corollary 4 (p. 206)] we have that $\psi_j, \phi_1, \ldots, \phi_k$ are linearly dependent, in particular $\psi_j \in \langle \phi_1, \ldots, \phi_k \rangle$ and thus we can write $\psi_i = \sum_{i=1}^k a_{ji} \phi_j$ for every $i = 1, \ldots, k$. Secondly, to check that we must have $a_{st} = a_{ts}$ for every $s, t = 1, \ldots, k$, we wedge the expression above by ϕ_i, and we obtain $\phi_i \wedge \psi_i = \sum_{i=1}^k a_{ji} \phi_i \wedge \phi_j$. Now, summing over all $i = 1, \ldots, k$ yields the expression that we know to be zero by hypothesis:

$$0 = \sum_{i=1}^{k} \phi_i \wedge \psi_i = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ji} \phi_i \wedge \phi_j,$$

now fixing $s, t = 1, \ldots, k$ we wedge with $\bigwedge_{i=1, i\neq s,t}^k \phi_i$ to obtain:

$$0 = \left(\sum_{i=1}^{k} \sum_{j=1}^{k} a_{ji} \phi_i \wedge \phi_j \right) \wedge \bigwedge_{i=1, i\neq s,t}^{k} \phi_i = a_{ts} (\phi_s \wedge \phi_t) \wedge \bigwedge_{i=1, i\neq s,t}^{k} \phi_i + a_{st} (\phi_t \wedge \phi_s) \wedge \bigwedge_{i=1, i\neq s,t}^{k} \phi_i$$

since again the wedge product is zero when we have repeating factors. Now since we wedged with something that was linearly independent, the only possible way in which this is zero is if and only if $0 = a_{ts} \phi_s \wedge \phi_t + a_{st} \phi_t \wedge \phi_s$, meaning that $0 = (a_{ts} - a_{st}) \phi_s \wedge \phi_t$ if we permute the wedge product, multiply by -1 and factor out the wedges. Since again ϕ_s, ϕ_t are linearly independent (and hence its wedge is nonzero), this means that $a_{ts} = a_{st}$. Since this reasoning follows for every $s, t = 1, \ldots, k$, we obtained the desired equality.

References

[1] M. Spivak, *A Comprehensive Introduction to Differential Geometry - Volume 1*, Publish or Perish INC., 2005.

[2] J. M. Lee, *Introduction to Smooth Manifolds*, Springer-Verlag, 2003.