Differential Geometry I - Homework 7

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April 7th, 2017

Exercise 1

Let α and β be closed differential forms on a manifold M.

1. Show that $\alpha \wedge \beta$ is closed:

   $$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge d\beta = 0 \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge 0 = 0,$$

   where we have used a standard equality of the differential of the wedge of two forms.

2. Show that if α is exact, say $d\nu = \alpha$, then $\alpha \wedge \beta$ is exact:

   $$\alpha \wedge \beta = d\nu \wedge \beta = d\nu \wedge \beta + (-1)^{\deg(\nu)}\nu \wedge 0 = d\nu \wedge \beta + (-1)^{\deg(\nu)}\nu \wedge d\beta = d(\nu \wedge \beta),$$

   where we again used the same equality of the differential of the wedge of two forms.

Exercise 2

1. We show that every closed 1-form in $\mathbb{S}^2$ is exact. For this, let w be a closed 1-form in $\mathbb{S}^2$. Let $N, S \in \mathbb{S}^2$ be the north pole and the south pole of the sphere. Consider w_N and w_S the restrictions of w to $\mathbb{S}^2 \setminus \{N\}$ and $\mathbb{S}^2 \setminus \{S\}$ respectively, which remain closed. Since we have that both $\mathbb{S}^2 \setminus \{N\}$ and $\mathbb{S}^2 \setminus \{S\}$ are diffeomorphic to $\mathbb{R}^2$ (as we proved in Homework 1), we obtain that the restrictions w_N and w_S are also closed when looking at them as forms over $\mathbb{R}^2$. By [2, Corollary 17.16 (p. 447)] we have that $H^1(\mathbb{R}^2) = 0$, thus we have 0-forms f_N and f_S on $\mathbb{R}^2$, that can be seen as forms on $\mathbb{S}^2 \setminus \{N\}$ and $\mathbb{S}^2 \setminus \{S\}$ respectively, such that $df_N = w_N$ and $df_S = w_S$. Moreover, on $\mathbb{S}^2 \setminus \{N\} \cap \mathbb{S}^2 \setminus \{S\} = \mathbb{S}^2 \setminus \{N, S\}$ we have:
$$d(f_N - f_S) = df_N - df_S = w_N - W_S = 0.$$
Now using using [2, Proposition 11.22 (p. 282)], we obtain that on $\mathbb{S}^2 \setminus \{N, S\}$ we have $f_N - f_S = r$ with $r \in \mathbb{R}$ is a fixed value. Thus define:
$$f(p) = \begin{cases} f_N(p) & \text{if} \quad p \in \mathbb{S}^2 \setminus \{N\} \\ f_S(p) + r & \text{if} \quad p \in \mathbb{S}^2 \setminus \{S\} \end{cases},$$
which is well defined because in the intersection the two values coincide by the above. Moreover, piece-wise this is smooth and it is smooth in the intersection, thus f is smooth, thus f is a 0-form. Finally, we have that:
$$df = \begin{cases} df_N = w_N & \text{in} \quad \mathbb{S}^2 \setminus \{N\} \\ d(f_S + r) = df_S = w_S & \text{in} \quad \mathbb{S}^2 \setminus \{S\} \end{cases},$$
and since w_N and w_S coincide in $\mathbb{S}^2 \setminus \{N, S\}$, we obtain that $df = w$, meaning that w is exact, as desired.

2. Prove that the following form is closed:
$$\sigma = \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} = \sigma_1 dy \wedge dz - \sigma_3 dx \wedge dz + \sigma_3 dx \wedge dy.$$
We note that when we apply the definition of the differential, the only terms that will survive are the ones that are proportional to $dx \wedge dy \wedge dz$, since the rest will have the wedge of two identical 1-forms, hence they will be zero. This means that:
$$\begin{aligned} d\sigma &= \frac{\partial \sigma_1}{\partial x} dx \wedge dy \wedge dz - \frac{\partial \sigma_2}{\partial y} dy \wedge dx \wedge dz + \frac{\partial \sigma_3}{\partial z} dz \wedge dx \wedge dy \\ &= \left(\frac{\partial \sigma_1}{\partial x} + \frac{\partial \sigma_2}{\partial y} + \frac{\partial \sigma_3}{\partial z} \right) dx \wedge dy \wedge dz. \end{aligned}$$
Thus we simply compute:
$$\begin{aligned} \frac{\partial \sigma_1}{\partial x} &= \frac{(x^2 + y^2 + z^2)^{3/2} - (3/2)x(x^2 + y^2 + z^2)^{1/2} 2x}{(x^2 + y^2 + z^2)^3} \\ &= \frac{(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3}(x^2 + y^2 + z^2 - 3x^2) = \frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}, \end{aligned}$$

$$\begin{aligned}\frac{\partial \sigma_2}{\partial y} &= \frac{(x^2+y^2+z^2)^{3/2} - (3/2)y(x^2+y^2+z^2)^{1/2}2y}{(x^2+y^2+z^2)^3} \\ &= \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}(x^2+y^2+z^2-3y^2) = \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{5/2}},\end{aligned}$$

$$\begin{aligned}\frac{\partial \sigma_3}{\partial x} &= \frac{(x^2+y^2+z^2)^{3/2} - (3/2)z(x^2+y^2+z^2)^{1/2}2z}{(x^2+y^2+z^2)^3} \\ &= \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}(x^2+y^2+z^2-3z^2) = \frac{-2z^2+x^2+y^2}{(x^2+y^2+z^2)^{5/2}}.\end{aligned}$$

Hence:

$$\begin{aligned}\frac{\partial \sigma_1}{\partial x} + \frac{\partial \sigma_2}{\partial y} + \frac{\partial \sigma_3}{\partial z} &= \frac{1}{(x^2+y^2+z^2)^{5/2}}(-2x^2+y^2+z^2 \\ &- \; 2y^2+x^2+z^2-2z^2+x^2+y^2) = 0,\end{aligned}$$

so $d\sigma = 0$ and σ is closed, as desired.

3. Evaluate the integral of σ over $\mathbb{S}^2$. We have:

$$\begin{aligned}\int_{\mathbb{S}^2} \sigma &= \int_{\mathbb{S}^2} \frac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2+y^2+z^2)^{3/2}} \\ &= \int_{\mathbb{S}^2} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\ &= \int_{\partial \mathbb{B}^3} xdy \wedge dz - ydx \wedge dz + zdx \wedge dy \\ &= \int_{\mathbb{B}^3} d(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy) \\ &= \int_{\mathbb{B}^3} 3dy \wedge dy \wedge dz = 3\mathrm{vol}(\mathbb{B}^3) = 3\frac{4}{3}\pi = 4\pi,\end{aligned}$$

where in the second equality we have used that $x^2+y^2+z^2 = 1$ on $\mathbb{S}^2$, on the third equality we have used that the boundary of the three dimensional closed ball of unit radius $\mathbb{B}^3$ is the two dimensional sphere of unit radius $\mathbb{S}^2$, on the fourth equality we have applied Stokes' Theorem, on the fourth equality we have differentiated using the definition and rearranging the terms to put the order of the 1-forms in such a way that we can add them up, on the fifth equality we notice that the integral of the ball over the three canonical coordinates equals the volume of such ball, and on the sixth equality we use that we know the value of the volume of any ball given its radius.

Suppose that σ were exact, that is, there is ν a 1-form with $d\nu = \sigma$. In that case, we can integrate:

$$\int_{\mathbb{S}^2} \sigma = \int_{\mathbb{S}^2} d\nu = \int_{\partial \mathbb{S}^2} \nu = 0,$$

where we used Stokes' Theorem for the second equality and the fact that $\partial\mathbb{S}^2 = \emptyset$ since the two dimensional sphere has no boundary, meaning that the integral must vanish. However, we computed above that this same integral was non-vanishing, hence σ exact yields a contradiction, meaning that σ cannot be exact, as desired.

Exercise 3

Let M be a compact orientable n-manifold without boundary, let θ be a $(n-1)$-form on M. We want to see that $d\theta$ is zero at some point of the manifold.

For this, we first note that we may restrict us to one of the connected components of M, where we will prove the existence of such a point. Equivalently, we may assume that M is connected. Now, notice that by Stokes' Theorem:

$$\int_M d\theta = \int_{\partial M} \theta = 0$$

because $\partial M = \emptyset$ by hypothesis. Moreover, since θ is a $(n-1)$-form, $d\theta$ is an n-form, and as such can be written as $d\theta = f dx^1 \wedge \cdots \wedge dx^n$ where f is a 0-form and $x^1, \ldots, x^n$ are the coordinates of a chart (x, U) on M. Suppose $f(p) \neq 0$ for every $p \in M$, say $f(q) > 0$ for a certain $q \in M$ (the case where $f(q) < 0$ for a certain $q \in M$ is completely analogous), we have two possibilities:

1. If $f(p) > 0$ for every $p \in M$, then we clearly have:

   $$\int_M d\theta = \int_M f dx^1 \wedge \cdots \wedge dx^n > 0,$$

   a contradiction with the above. Hence this cannot happen.

2. If there is $r \in M$ with $f(r) < 0$, then notice that $f : M \longrightarrow \mathbb{R}$ is a smooth map (in particular continuous) from a connected topological space M to $\mathbb{R}$ a totally ordered set equipped with the order topology. Thus we can apply the n-dimensional version of the Intermediate Value Theorem to say that the whole range from $f(r)$ to $f(q)$ is contained in the image of f, that is, $[f(r), f(q)] \in \text{im}(f)$. In particular, there is a point $s \in M$ with $f(s) = 0$, meaning that $d\theta$ at this $s \in M$ must be zero, as desired.

Exercise 4

1. Let M be a compact oriented n-manifold with non-empty boundary. We want to show that there is no retraction $\phi : M \longrightarrow \partial M$.

   For this, let $w \in \Omega^{n-1}(\partial M)$ be an $(n-1)$-form in ∂M whose integral over ∂M is strictly positive. Notice that we can guarantee its existence because in virtue of [2, Theorem 17.30 (p. 454)] we have that the integral of the corresponding forms gives an isomorphism $H_c^{n-1} \cong \mathbb{R}$, thus given a positive real number we can take the isomorphism to obtain the cohomology class of certain form, and this form will have integral the same positive real number we started with. Consider now $\beta = \phi^* w$ a $(n-1)$-form on M and integrate $d\beta$ over M in the two natural ways:

$$0 = \int_M \phi^*(dw) = \int_M d(\phi^* w) = \int_M d\beta = \int_{\partial M} \beta = \int_{\partial M} \phi^* w = \int_{\partial M} w > 0.$$

   Starting from the integral of $d\beta$ over M to the left, we used the definition of β, then we used that the pullback and the exterior derivative commute in virtue of [2, Proposition 14.26 (p. 366)] and finally we used that $dw \in \Omega^n(\partial M) = \{0\}$ because ∂M is $(n-1)$-dimensional, thus all the n-forms are zero. Starting from the integral of $d\beta$ over M to the right, we used Stokes' Theorem, then we used the definition of β, then we used that ϕ restricted to ∂M is the identity and finally we used the construction of w, namely that it has strictly positive integral. Since this is an obvious contradiction, we have that such a retraction $\phi : M \longrightarrow \partial M$ cannot exist, as desired.

2. We want to prove that if $F : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ is a smooth map from the closed n-dimensional unit ball to itself, then F has a fixed point.

   To prove this, suppose F as above has no fixed points, that is, $F(x) \neq x$ for every $x \in \mathbb{B}^n$. Then given $x \in M$, consider the half line starting at $F(x)$ and intersecting x, namely the one parametrized by $L(x,t) = F(x)(1-t) + xt$ with $t \in (0, \infty)$. Since $F(x) \neq x$, such a half line always exists and thus it is well defined. We note that since both $F(x), x \in \mathbb{B}^n$, this half line must intersect $\partial\mathbb{B}^n$ for some time, say t_x, and such an intersection always occurs once (because it is only a half line, we do not go backwards in time, and we do not pick time $t = 0$: this is really important as we will see below because if $x \in \partial\mathbb{B}^n$ nothing stops us from having $F(x) \in \partial\mathbb{B}^n$, but by not taking $t = 0$ we make the half line so that $F(x) \notin L(x,t)$ and thus the intersection of this half line with the boundary is exactly one point). We define $\phi : \mathbb{B}^n \longrightarrow \partial\mathbb{B}^n$ as $\phi(x) = L(x, t_x)$, that is, the unique point in $L(x, t_x) \cap \partial\mathbb{B}^n$, which clearly is well defined. Moreover, this function is smooth since:

$$\begin{aligned}
\frac{\partial\phi}{\partial x}(x) &= \lim_{h\to 0}\frac{1}{h}(L_{x+h}(t_{x+h}) - L_x(t_x)) \\
&= \lim_{h\to 0}\frac{1}{h}(L_{x+h}(t_{x+h}) - L_x(t_{x+h}) + L_x(t_{x+h}) - L_x(t_x)) \\
&= \lim_{h\to 0}\frac{1}{h}(L_{x+h}(t_{x+h}) - L_x(t_{x+h})) + \lim_{h\to 0}\frac{1}{h}(L_x(t_{x+h}) - L_x(t_x)) \\
&= \frac{\partial L}{\partial x}(x, t_x + h) + \frac{\partial L}{\partial t}(x, t_x),
\end{aligned}$$

and this is smooth because $L(x,t)$ is smooth in both variables. Thus we showed that ϕ can be differentiated once and that its differential $\partial\phi/\partial x$ is smooth, hence ϕ is smooth. Moreover, if $y \in \partial\mathbb{B}^n$ then we have that $L(y,1) = y \in \partial\mathbb{B}^n$, and since we argued above that the intersection point of the line is unique, we have that $t_y = 1$ and $\phi(y) = y$, so ϕ is the identity map in $\partial\mathbb{B}^n$. However, the ϕ we now constructed is a smooth retraction satisfying the conditions in the above section, which is a contradiction. Hence any $F : \mathbb{B}^n \longrightarrow \mathbb{B}^n$ must have one fixed point, as desired.

Exercise 5

Let $f : M^k \longrightarrow \mathbb{R}^n$, $g : N^l \longrightarrow \mathbb{R}^n$ be smooth, M, N compact oriented manifolds with $n = k + l + 1$ and $f(M) \cap g(N) = \emptyset$. We define $\alpha_{f,g} : M \times N \longrightarrow \mathbb{S}^{n-1}$ by $\alpha_{f,g}(p, q) = (g(q) - f(p))/|g(q) - f(p)|$ and $\ell(f, g) = \deg(\alpha_{f,g})$.

1. We want to prove that $\ell(f, g) = (-1)^{(k+1)(l+1)}\ell(g, f)$. Note that if we define:

$$\begin{array}{rcrcl} A & : & \mathbb{S}^{n-1} & \longrightarrow & \mathbb{S}^{n-1} \\ & & x & \longmapsto & -x \end{array}, \qquad \begin{array}{rcrcl} T & : & N \times M & \longrightarrow & M \times N \\ & & (q,p) & \longmapsto & (p,q) \end{array},$$

then we have that:

$$\begin{aligned} A \circ \alpha_{f,g} \circ T(q, p) &= A \circ \alpha_{f,g}(p, q) = A\left(\frac{g(q) - f(p)}{|g(q) - f(p)|}\right) \\ &= -\frac{g(q) - f(p)}{|g(q) - f(p)|} = \frac{f(p) - g(q)}{|f(p) - g(q)|} = \alpha_{g,f}(q, p), \end{aligned}$$

meaning that $A \circ \alpha_{f,g} \circ T = \alpha_{g,f}$ as functions from $N \times M$ to $\mathbb{S}^{n-1}$. Now, let (x, U), (y, V) be charts on M, N respectively, notice that:

$$\int_{N\times M} T^*(dx^1 \wedge \cdots \wedge dx^k \wedge dy^1 \wedge \cdots \wedge dy^l) = \int_{M\times N} d(x^1 \circ T) \wedge$$
$$\cdots \wedge d(x^k \circ T) \wedge d(y^1 \circ T) \wedge \cdots \wedge d(y^l \circ T) = \int_{M\times N} dy^1 \wedge$$
$$\cdots \wedge dy^l \wedge dx^1 \wedge \cdots \wedge dx^k = (-1)^{kl} \int_{M\times N} dx^1 \wedge \cdots \wedge dx^k \wedge dy^1 \wedge \cdots \wedge dy^l,$$

where we have used [2, Lemma 14.16 (p. 361)]. This proves that $\deg(T) = (-1)^{kl}$. Moreover, by the discussion in [1, p. 277] we know that $\deg(A) = (-1)^{k+l+1-1} = (-1)^{k+l}$. Finally, using [2, Proposition 17.36 (p. 459)] we have that:

$$\begin{aligned} \ell(g, f) &= \deg(\alpha_{g,f}) = \deg(A) \deg(\alpha_{f,g}) \deg(T) \\ &= (-1)^{k+l+1}(-1)^{kl+1} \deg(\alpha_{f,g}) = (-1)^{(k+1)(l+1)}\ell(f, g), \end{aligned}$$

so multiplying by $(-1)^{(k+1)(l+1)}$ we obtain that $\ell(f, g) = (-1)^{(k+1)(l+1)}\ell(g, f)$, the desired result.

2. Given homotopies H between f, $\overline{f}$ and K between g, $\overline{g}$, we clearly have:

$$\begin{array}{rcrcl} h & : & M \times N \times [0, 1] & \longrightarrow & \mathbb{S}^{n-1} \\ & & (p, q, t) & \longmapsto & \frac{g(q)-H(p,t)}{|g(q)-H(p,t)|} \end{array}, \quad \begin{array}{rcrcl} k & : & M \times N \times [0, 1] & \longrightarrow & \mathbb{S}^{n-1} \\ & & (p, q, t) & \longmapsto & \frac{K(q,t)-\overline{f}(p)}{|K(q,t)-\overline{f}(p)|} \end{array},$$

that are both smooth as composition of smooth functions, and the fact that $\{H(p, t) : p \in M\} \cap \{K(q, t) : q \in N\} = \emptyset$ for every $t \in [0, 1]$ guarantees that they are well defined and indeed we can divide. Moreover, notice how:

$$h(p,q,0) = \frac{g(q) - H(p,0)}{|g(q) - H(p,0)|} = \alpha_{f,g}(p,q),\ k(p,q,0) = \frac{K(q,0) - \overline{f}(p)}{|K(q,0) - \overline{f}(p)|} = \alpha_{\overline{f},g}(p,q),$$

$$h(p,q,1) = \frac{g(q) - H(p,1)}{|g(q) - H(p,1)|} = \alpha_{\overline{f},g}(p,q),\ k(p,q,1) = \frac{K(q,1) - \overline{f}(p)}{|K(q,1) - \overline{f}(p)|} = \alpha_{\overline{f},\overline{g}}(p,q),$$

hence h is a smooth homotopy between $\alpha_{f,g}$, $\alpha_{\overline{f},g}$ and k between is a smooth homotopy between $\alpha_{\overline{f},g}$, $\alpha_{\overline{f},\overline{g}}$. Thus applying [2, Proposition 17.36 (p. 459)] we have that if two smooth maps are homotopic, then they have the same degree, hence $\ell(f,g) = \deg(\alpha_{f,g}) = \deg(\alpha_{\overline{f},g}) = \deg(\alpha_{\overline{f},\overline{g}}) = \ell(\overline{f},\overline{g})$, what we wanted.

3. For $f, g : \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ we want to explicitly compute $\ell(f,g)$. For this, we will use [2, Theorem 17.35 (p. 457)], that assures us of:

$$\deg(\alpha_{f,g}) \int_{\mathbb{S}^2} w = \int_{S^1 \times S^1} \alpha^*_{f,g}(w)$$

where w is the differential form of our choice, the only condition is that its integral is non zero. Before integrating everything, we make some tweaks to our functions and our coordinate systems so that we will obtain the desired result. First, since $\mathbb{S}^1$ is diffeomorphic to the unit interval by gluing 0 and 1, we will take the functions $f, g : \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ as functions $f, g : [0,1] \longrightarrow \mathbb{R}^3$ with $f(0) = f(1)$ and $g(0) = g(1)$, and we name their coordinates u, v respectively. Second, we notice that $\mathbb{S}^2 \subset \mathbb{R}^3$ so we may take Cartesian coordinates for our choice of w we look at Exercise 2 above and choose $w = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$, that we already know that integrates to 4π, obtaining that:

$$\deg(\alpha_{f,g}) = \frac{1}{4\pi} \int_{S^1 \times S^1} \alpha^*_{f,g}(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy).$$

Once we are here, the pain begins. Since $f, g : \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ we may write them in components as $(f_1(u), f_2(u), f_3(u))$, $(g_1(v), g_2(v), g_3(v))$ respectively. Since $\alpha_{f,g} : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{R}^3$, by looking at its definition we can write the components of $\alpha_{f,g}$ as $(\alpha_1(u,v), \alpha_2(u,v), \alpha_3(u,v))/r(u,v)$ with $\alpha_i(u,v) = g_i(v) - f_i(u)$ for $i = 1,2,3$ and $r(u,v) = |g(v) - f(u)|$. Equipped with these horrible expressions, we can use [2, Lemma 14.16 (p. 361)] to explicitly compute the pullback:

$$\alpha^*_{f,g}(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy) = (x \circ \alpha_{f,g}) d(y \circ \alpha_{f,g}) \wedge d(z \circ \alpha_{f,g})$$
$$-(y \circ \alpha_{f,g}) d(x \circ \alpha_{f,g}) \wedge d(z \circ \alpha_{f,g}) + (z \circ \alpha_{f,g}) d(x \circ \alpha_{f,g}) \wedge d(y \circ \alpha_{f,g})$$

so:

$$\begin{aligned} \alpha^*_{f,g}(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy) &= \frac{\alpha_1(u,v)}{r(u,v)} d\left(\frac{\alpha_2(u,v)}{r(u,v)}\right) \wedge d\left(\frac{\alpha_3(u,v)}{r(u,v)}\right) \\ &- \frac{\alpha_2(u,v)}{r(u,v)} d\left(\frac{\alpha_1(u,v)}{r(u,v)}\right) \wedge d\left(\frac{\alpha_3(u,v)}{r(u,v)}\right) \\ &+ \frac{\alpha_3(u,v)}{r(u,v)} d\left(\frac{\alpha_1(u,v)}{r(u,v)}\right) \wedge d\left(\frac{\alpha_2(u,v)}{r(u,v)}\right). \end{aligned}$$

Now, we have to expand this madness and rearrange the terms to integrate along $du \wedge dv$. To do so, we notice that we have $r(u,v)$ dividing everywhere, and the expressions look very symmetric. In fact, the way to proceed is to explicitly differentiate $\alpha_i(u,v)$ for $i = 1,2,3$ to obtain the derivatives of $f_i(u)$ and $g_i(v)$ for $i = 1,2,3$, but whenever we encounter the derivative of $1/r(u,v)$, we do not expand it and leave it formally. It turns out that all the terms with those derivatives cancel out, but since we want to preserve both ours and the reader's mental health, we will not present the computation here. Thus a posteriori we can apply the product rule by taking the term $1/r(u,v)$ as constant, meaning that in fact we can factor it out to obtain:

$$\begin{aligned}\alpha^*_{f,g}(w) &= \frac{\alpha_1(u,v)}{r(u,v)^3} d(\alpha_2(u,v)) \wedge d(\alpha_3(u,v)) \\ &- \frac{\alpha_2(u,v)}{r(u,v)^3} d(\alpha_1(u,v)) \wedge d(\alpha_3(u,v)) \\ &+ \frac{\alpha_3(u,v)}{r(u,v)^3} d(\alpha_1(u,v)) \wedge d(\alpha_2(u,v)).\end{aligned}$$

Hence what we have proven so far is that:

$$\deg(\alpha_{f,g}) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\nu}{r(u,v)^3}$$

where:

$$\begin{aligned}\nu &= \alpha_1(u,v) d(\alpha_2(u,v)) \wedge d(\alpha_3(u,v)) \\ &- \alpha_2(u,v) d(\alpha_1(u,v)) \wedge d(\alpha_3(u,v)) \\ &+ \alpha_3(u,v) d(\alpha_1(u,v)) \wedge d(\alpha_2(u,v)).\end{aligned}$$

This is some progress, but the calculations remain more than tedious. Bracing ourselves, we start computing: differentiating using the discussion on [2, p. 363] and selecting only the crossed terms of the form $du \wedge dv$ and $dv \wedge du$. The ensuing mess of summands can only be described as the closest thing to chaos that we have ever experienced, we fell in a seeming bottomless hole of terms that tried to suck us in and never let us go. The details will be spared since if we stare at the abyss for too long, the abyss eventually stares back, a risk that we need to avoid at all costs. Somehow, we managed to see the light and simplify the monstrosity into a more bearable freak[1] (in the sign given by $du \wedge dv$):

$$\begin{aligned}&-f_1'(u)g_2'(v)g_3(v) + f_1'(u)g_2'(v)f_3(u) - f_1'(u)g_3'(v)g_2(v) + f_1'(u)g_3'(v)f_2(u) \\ &-g_1'(v)f_2'(u)g_3(v) + g_1'(v)f_2'(u)f_3(u) - g_1'(v)f_3'(u)g_2(v) + g_1'(v)f_3'(u)f_2(u) \\ &+g_1(u)f_2'(u)g_3'(v) + f_1(u)f_2'(u)g_3'(v) - g_1(u)f_3'(u)g_2'(v) + f_1(u)f_3'(u)g_2'(v)\end{aligned}$$

[1]Since there are so many signs in the expression below, it is possible that some of them were transcribed as negative when they were positive and vice versa. We apologize if it is the case. However, the final result has been checked multiple times to ensure that it is correct.

almost inexplicably, this can be simplified to:

$$f_1'(u)(g_2'(v)(g_3(v) - f_3(u)) + g_3'(v)(g_2(v) - f_2(u))) - g_1'(v)(f_2'(u)(g_3(v) \\ -f_3(u)) + f_3'(u)(g_2(v) - f_2(u))) + (g_1(u) - f_1(u))(f_2'(u)g_3'(v) + f_3'(u)g_2'(v))$$

which is precisely the determinant of the following matrix, but with sign changed:

$$\begin{bmatrix} f_1'(u) & f_2'(u) & f_3'(u) \\ g_1'(v) & g_2'(v) & g_3'(v) \\ g_1(v) - f_1(u) & g_2(v) - f_2(u) & g_3(v) - f_3(u) \end{bmatrix}$$

thus:

$$\deg(\alpha_{f,g}) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{-A(u,v) du \wedge dv}{r(u,v)^3} = \frac{-1}{4\pi} \int_0^1 \int_0^1 \frac{A(u,v)}{r(u,v)^3},$$

precisely what we wanted to obtain.

4. We want to show that if f, g are coplanar, then $\ell(f,g) = 0$.

   First, we suppose f, g are coplanar in the first two coordinates. This means that instead of having the image of $\alpha_{f,g}$ contained in $\mathbb{S}^{n-1} = \{(x_0, \ldots, x_{n-1}) : x_0^2 + \cdots + x_{n-1}^2 = 1\}$, we actually have that the image of $\alpha_{f,g}$ is contained in $\{(x_0, \ldots, x_{n-1}) : x_0^2 + x_1^2 = 1, x_i = 0 \text{ if } i \neq 0, 1\}$. In virtue of the discussion on [2, p. 78], we have that the rank of $D_{\alpha_{f,g}}$ the differential of $\alpha_{f,g}$ is less than or equal than 1, the maximum dimension of the image of $\alpha_{f,g}$. In particular $\text{rank}(D_{\alpha_{f,g}}) \leq 1 < n-1$ assuming that M, N have both at least dimension 1. Since then $D_{\alpha_{f,g}}$ is a $(n-1) \times (n-1)$ matrix with rank strictly less than $n-1$, this means that $J_{\alpha_{f,g}} = \det(D_{\alpha_{f,g}}) = 0$, that is, the Jacobian of $\alpha_{f,g}$ is zero. Now letting dV being a volume form in $\mathbb{S}^{n-1}$ and using [2, Proposition 14.20 (p. 361)] we have that $\alpha_{f,g}^*(dV) = 0$ because it has multiplying the Jacobian of $\alpha_{f,g}$, which we just saw is zero. Finally, using the definition of the degree of $\alpha_{f,g}$, that is using [2, Theorem 17.35 (p. 457)] we obtain that:

$$\deg(\alpha_{f,g}) = \frac{\int_{M \times N} \alpha_{f,g}^*(dV)}{\int_{\mathbb{S}^{n-1}} dV} = 0$$

   because the numerator is zero. Thus $\ell(f,g) = 0$, as desired.

   Now, suppose f, g are coplanar in a general plane. Then by the usual completion of a basis of a vector space, we may take coordinates in $\mathbb{R}^n$ so that this plane is precisely defined by the first and second components. Thus applying the above we obtain that indeed $\ell(f,g) = 0$, as desired.

References

[1] M. Spivak, *A Comprehensive Introduction to Differential Geometry - Volume 1*, Publish or Perish INC., 2005.

[2] J. M. Lee, *Introduction to Smooth Manifolds (Second Edition)*, Springer-Verlag, 2013.