Differential Geometry I - Homework 8

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Exercise 1

Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by $f(x,y) = (\cos(x), \sin(x), \cos(y), \sin(y))$.

1. We have that f is an immersion, because:

$$Df = \begin{bmatrix} -\sin(x) & 0 \\ \cos(x) & 0 \\ 0 & -\sin(y) \\ 0 & \cos(y) \end{bmatrix}$$

and thus considering the following minors and their corresponding determinants:

$$\begin{aligned} M_1 &= \begin{bmatrix} -\sin(x) & 0 \\ 0 & -\sin(y) \end{bmatrix}, \quad \det(M_1) = \sin(x)\sin(y) \\ M_2 &= \begin{bmatrix} \cos(x) & 0 \\ 0 & \cos(y) \end{bmatrix}, \quad \det(M_1) = \cos(x)\cos(y) \\ M_3 &= \begin{bmatrix} \cos(x) & 0 \\ 0 & -\sin(y) \end{bmatrix}, \quad \det(M_1) = -\cos(x)\sin(y) \\ M_4 &= \begin{bmatrix} -\sin(x) & 0 \\ 0 & \cos(y) \end{bmatrix}, \quad \det(M_1) = -\sin(x)\cos(y) \end{aligned}$$

so we have:

$$\begin{aligned} \det(M_1) &= 0 \iff x = n\pi \text{ or } y = n\pi \text{ for some } n \in \mathbb{Z} \\ \det(M_2) &= 0 \iff x = n\pi - \pi/2 \text{ or } y = n\pi - \pi/2 \text{ for some } n \in \mathbb{Z}. \end{aligned}$$

Thus the points having $\det(M_1) = 0 = \det(M_2)$ are of the form $(n\pi, k\pi - \pi/2)$ or $(k\pi - \pi/2, n\pi)$ with $n, k \in \mathbb{Z}$. If $(x,y) \in \mathbb{R}^2$ is not of this form, we have a minor with non zero determinant, hence $\text{rank}(Df) = 2$. Moreover for $n, k \in \mathbb{Z}$ we have:

$$\begin{aligned} (n\pi, k\pi - \pi/2) &\quad \text{has} \quad \det(M_3) \neq 0 \\ (k\pi - \pi/2, n\pi) &\quad \text{has} \quad \det(M_4) \neq 0, \end{aligned}$$

so in both cases we have a minor with non zero determinant. Hence for every $(x,y) \in \mathbb{R}^2$ we have a minor with non zero determinant, hence $\text{rank}(Df) = 2$ and by the discussion in [2, p. 78] we have that f is an immersion.

Now we observe that since the sine and cosine functions are periodic, to obtain $f(\mathbb{R}^2)$ is enough to have $x, y \in [0, 2\pi)$. Moreover, for each pair these values we have a unique point in $f(\mathbb{R}^2)$, namely $(\cos(x), \sin(x), \cos(y), \sin(y))$. Hence $f(\mathbb{R}^2) = \{(\cos(x), \sin(x), \cos(y), \sin(y)) : x, y \in [0, 2\pi)\}$. Consider the function $\psi : f(\mathbb{R}^2) \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ defined by $\psi(\cos(x), \sin(x), \cos(y), \sin(y)) = (\cos(x), \sin(x), \cos(y), \sin(y))$, which is clearly well defined since $\mathbb{S}^1 = \{(\cos(t), \sin(t)) : t \in [0, 2\pi)\}$ in virtue of the polar coordinates. This function is continuous as composition of continuous functions. Moreover, since both $f(\mathbb{R}^2)$ and $\mathbb{S}^1 \times \mathbb{S}^1$ can be written as the same set, and this function is a bijective identification between those sets (namely the identity), ψ is both injective and surjective. Finally, the function $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow f(\mathbb{R}^2)$ defined as $\phi(\cos(u), \sin(u), \cos(v), \sin(v)) = (\cos(u), \sin(u), \cos(v), \sin(v))$ is well defined and continuous by the same reasons that apply to ψ. Clearly we have $\phi \circ \psi = \mathrm{id}_{f(\mathbb{R}^2)}$ and $\psi \circ \phi = \mathrm{id}_{\mathbb{S}^1 \times \mathbb{S}^1}$, so ψ is indeed a homeomorphism. Thus $f(\mathbb{R}^2) \cong \mathbb{S}^1 \times \mathbb{S}^1$, which is a torus.

2. We consider the frame in $\mathbb{R}^2$ given by $e_1 = \partial/\partial x$ and $e_2 = \partial/\partial y$. To prove that it is orthonormal in $f(\mathbb{R}^2)$ with the induced metric, we compute:

$$
\begin{aligned}
\langle e_1, e_1 \rangle_{f(\mathbb{R}^2)} &= \langle f_*(e_1), f_*(e_1) \rangle_{\mathbb{R}^4} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle \\
&= \langle (-\sin(x), \cos(x), 0, 0), (-\sin(x), \cos(x), 0, 0) \rangle \\
&= \sin(x)^2 + \cos(x)^2 = 1 \\
\langle e_2, e_2 \rangle_{f(\mathbb{R}^2)} &= \langle f_*(e_2), f_*(e_2) \rangle_{\mathbb{R}^4} = \left\langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \right\rangle \\
&= \langle (0, 0, -\sin(y), \cos(y)), (0, 0, -\sin(y), \cos(y)) \rangle \\
&= \sin(y)^2 + \cos(y)^2 = 1 \\
\langle e_1, e_2 \rangle_{f(\mathbb{R}^2)} &= \langle f_*(e_1), f_*(e_2) \rangle_{\mathbb{R}^4} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\
&= \langle (-\sin(x), \cos(x), 0, 0), (0, 0, -\sin(y), \cos(y)) \rangle = 0 \\
\langle e_2, e_1 \rangle_{f(\mathbb{R}^2)} &= \langle e_1, e_2 \rangle_{f(\mathbb{R}^2)} = 0,
\end{aligned}
$$

hence the push-forwards indeed form an orthonormal frame, as desired. We clearly have that $w_1 = dx$ and $w_2 = dy$ is a coframe, meaning that $dw_1 = 0$ and $dw_2 = 0$ and since $dw_1 = w_{12} \wedge w_2$ and $dw_2 = -w_{12} \wedge w_1$, that is, the connection form has those coefficients as components, we must thus have $w_{12} = 0$.

3. Since the Gaussian curvature K satisfies $dw_{12} = -K w_1 \wedge w_2$, having $dw_{12} = 0$ implies $K = 0$.

Exercise 2

Consider $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, we define for $(x,y) \in \mathbb{H}^2$ and $u, v \in T_{(x,y)}\mathbb{H}^2$ the inner product $\langle u, v\rangle_{(x,y)} = \langle u, v\rangle_{\mathbb{R}^2}/y^2$ via the standard inner product on $\mathbb{R}^2$.

First, this is a Riemannian metric since in the notation above:

$$\begin{aligned}
\langle u, v\rangle_{(x,y)} &= \frac{\langle u, v\rangle_{\mathbb{R}^2}}{y^2} = \frac{\langle v, u\rangle_{\mathbb{R}^2}}{y^2} = \langle v, u\rangle_{(x,y)} \\
\langle v, v\rangle_{(x,y)} &= \frac{\langle v, v\rangle_{\mathbb{R}^2}}{y^2} \geq 0 \\
\langle v, v\rangle_{(x,y)} &= 0 \iff \langle v, v\rangle_{\mathbb{R}^2} = 0 \iff v = 0,
\end{aligned}$$

where for the first we used the symmetry of the standard inner product on $\mathbb{R}^2$, on the second we used that both $\langle u \cdot v\rangle_{\mathbb{R}^2}$ and y^2 are positive and on the third we used that the standard inner product defines a Riemannian metric on $\mathbb{R}^2$.

Second, we can compute the curvature considering the frame in $\mathbb{R}^2$ given by $e_1 = y\partial/\partial x$ and $e_2 = y\partial/\partial y$. To prove that it is orthonormal in $\mathbb{H}^2$ with the above metric, we compute:

$$\begin{aligned}
\langle e_1, e_1\rangle_{(x,y)} &= \frac{\langle y\partial/\partial x, y\partial/\partial x\rangle_{\mathbb{R}^2}}{y^2} = \frac{y^2}{y^2}\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle = 1 \\
\langle e_2, e_2\rangle_{(x,y)} &= \frac{\langle y\partial/\partial y, y\partial/\partial y\rangle_{\mathbb{R}^2}}{y^2} = \frac{y^2}{y^2}\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle = 1 \\
\langle e_1, e_2\rangle_{(x,y)} &= \frac{\langle y\partial/\partial x, y\partial/\partial y\rangle_{\mathbb{R}^2}}{y^2} = \frac{y^2}{y^2}\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\rangle = 0 \\
\langle e_2, e_1\rangle_{(x,y)} &= \langle e_1, e_2\rangle_{(x,y)} = 0,
\end{aligned}$$

where we have used that $\partial/\partial x$ and $\partial/\partial y$ form an orthonormal basis of $\mathbb{R}^2$ with the standard metric. Hence e_1 and e_2 indeed form an orthonormal frame, as desired. We clearly have that $w_1 = (1/y)dx$ and $w_2 = (1/y)dy$ is a coframe, meaning that:

$$\begin{aligned}
dw_1 &= \frac{-1}{y^2} dy \wedge dx = \left(\frac{1}{y}dx\right) \wedge \left(\frac{1}{y}dy\right) = \left(\frac{1}{y}dx\right) \wedge w_2 \\
dw_2 &= 0,
\end{aligned}$$

thus since $dw_1 = w_{12} \wedge w_2$ and $dw_2 = -w_{12} \wedge w_1$, we have:

$$\begin{aligned}
w_{12} &= \frac{1}{y}dx \\
dw_{12} &= \frac{-1}{y^2}dy \wedge dx = -(-1)\left(\frac{1}{y}dx\right) \wedge \left(\frac{1}{y}dy\right) = -(-1)w_1 \wedge w_2
\end{aligned}$$

so $K = -1$, as desired.

Exercise 3

Consider $\mathbb{R}^2$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ a strictly positive smooth function. We define for $(x, y) \in \mathbb{R}^2$ and $u, v \in T_{(x,y)}\mathbb{R}^2$ the inner product $\langle u, v\rangle_{(x,y)} = \langle u, v\rangle_{\mathbb{R}^2}/g(x, y)^2$ via the standard inner product on $\mathbb{R}^2$.

We can compute the curvature considering the frame in $\mathbb{R}^2$ given by $e_1 = g(x, y)\partial/\partial x$ and $e_2 = g(x, y)\partial/\partial y$. To prove that it is orthonormal in $\mathbb{H}^2$ with the above metric, we compute:

$$\begin{aligned}
\langle e_1, e_1\rangle_{(x,y)} &= \frac{\langle g(x,y)\partial/\partial x, g(x,y)\partial/\partial x\rangle_{\mathbb{R}^2}}{g(x,y)^2} = \frac{g(x,y)^2}{g(x,y)^2}\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right\rangle = 1\\
\langle e_2, e_2\rangle_{(x,y)} &= \frac{\langle g(x,y)\partial/\partial y, g(x,y)\partial/\partial y\rangle_{\mathbb{R}^2}}{g(x,y)^2} = \frac{g(x,y)^2}{g(x,y)^2}\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right\rangle = 1\\
\langle e_1, e_2\rangle_{(x,y)} &= \frac{\langle g(x,y)\partial/\partial x, g(x,y)\partial/\partial y\rangle_{\mathbb{R}^2}}{g(x,y)^2} = \frac{g(x,y)^2}{g(x,y)^2}\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\rangle = 0\\
\langle e_2, e_1\rangle_{(x,y)} &= \langle e_1, e_2\rangle_{(x,y)} = 0,
\end{aligned}$$

where we have used that $\partial/\partial x$ and $\partial/\partial y$ form an orthonormal basis of $\mathbb{R}^2$ with the standard metric. Hence e_1 and e_2 indeed form an orthonormal frame, as desired. We clearly have that $w_1 = (1/g(x, y))dx$ and $w_2 = (1/g(x, y))dy$ is a coframe. From now on, we omit the variables where we consider g to not overload the notation. We have:

$$\begin{aligned}
dw_1 &= \frac{-1}{g^2}\frac{\partial g}{\partial y}dy \wedge dx = \left(\frac{1}{g}\frac{\partial g}{\partial y}dx\right) \wedge \left(\frac{1}{g}dy\right) = \left(\frac{1}{g}\frac{\partial g}{\partial y}dx\right) \wedge w_2\\
dw_2 &= \frac{-1}{g^2}\frac{\partial g}{\partial x}dx \wedge dy = \left(\frac{1}{g}\frac{\partial g}{\partial x}dy\right) \wedge \left(\frac{1}{g}dx\right) = \left(\frac{1}{g}\frac{\partial g}{\partial x}dy\right) \wedge w_1,
\end{aligned}$$

thus since $dw_1 = w_{12} \wedge w_2$ and $dw_2 = -w_{12} \wedge w_1$, we have:

$$\begin{aligned}
w_{12} &= \frac{1}{g}\frac{\partial g}{\partial y}dx - \frac{1}{g}\frac{\partial g}{\partial x}dy\\
dw_{12} &= \frac{\partial}{\partial y}\left(\frac{1}{g}\frac{\partial g}{\partial y}\right) dy \wedge dx - \frac{\partial}{\partial y}\left(\frac{1}{g}\frac{\partial g}{\partial x}\right) dx \wedge dy\\
&= \left(\frac{-1}{g^2}\frac{\partial g}{\partial y}\frac{\partial g}{\partial y} + \frac{1}{g}\frac{\partial^2 g}{\partial y^2}\right) dy \wedge dx - \left(\frac{-1}{g^2}\frac{\partial g}{\partial x}\frac{\partial g}{\partial x} + \frac{1}{g}\frac{\partial^2 g}{\partial x^2}\right) dx \wedge dy\\
&= -\left(-\left(\frac{\partial g}{\partial y}\right)^2 + g\frac{\partial^2 g}{\partial y^2}\right)\left(\frac{1}{g}dx\right) \wedge \left(\frac{1}{g}dy\right) - \left(-\left(\frac{\partial g}{\partial x}\right)^2 + g\frac{\partial^2 g}{\partial x^2}\right)\left(\frac{1}{g}dx\right) \wedge \left(\frac{1}{g}dy\right)\\
&= -\left(g\left(\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial x^2}\right) - \left(\left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial x}\right)^2\right)\right) w_1 \wedge w_2
\end{aligned}$$

so:

$$K = g\left(\frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial x^2}\right) - \left(\left(\frac{\partial g}{\partial y}\right)^2 + \left(\frac{\partial g}{\partial x}\right)^2\right)$$

as desired.

Exercise 4

Let $\mathbb{S}^2$ be the unit sphere inside $\mathbb{R}^3$ with the induced metric from the latter.

1. The antipodal map $A : \mathbb{S}^2 \longrightarrow \mathbb{S}^{\nvDash}$ given by $A(x, y, z) = (-x, -y, -z)$ is an isometry because for $(x_1, y_1, z_1), (x_1, y_1, z_1) \in \mathbb{S}^2$ we have:

$$\begin{aligned} \langle A(x_1, y_1, z_1), A(x_2, y_2, z_2)\rangle &= \langle(-x_1, -y_1, -z_1), (-x_1, -y_1, -z_1)\rangle \\ &= x_1x_2 + y_1y_2 + z_1z_2 = \langle(x_1, y_1, z_1), (x_1, y_1, z_1)\rangle. \end{aligned}$$

2. To compute the curvature of the sphere we consider its parametrization given by $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined as $f(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ and the frame in $\mathbb{R}^2$ given by $e_1 = \partial/\partial\theta$ and $e_2 = (1/\sin(\theta))\partial/\partial\phi$, in an appropriate open where it is defined. To prove that it is orthonormal in $f(\mathbb{R}^2)$ with the induced metric, we compute:

$$\begin{aligned}
\langle e_1, e_1\rangle_{f(\mathbb{R}^2)} &= \langle f_*(e_1), f_*(e_1)\rangle_{\mathbb{R}^3} = \left\langle \frac{\partial f}{\partial\theta}, \frac{\partial f}{\partial\theta} \right\rangle \\
&= \langle(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta)), \\
&\quad (\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta))\rangle \\
&= \cos(\theta)^2\sin(\phi)^2 + \sin(\phi)^2\cos(\theta)^2 + \sin(\theta)^2 = 1 \\
\langle e_2, e_2\rangle_{f(\mathbb{R}^2)} &= \langle f_*(e_2), f_*(e_2)\rangle_{\mathbb{R}^3} = \frac{1}{\sin(\theta)^2}\left\langle \frac{\partial f}{\partial\phi}, \frac{\partial f}{\partial y} \right\rangle \\
&= \frac{1}{\sin(\theta)^2}\langle(-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0), \\
&\quad (-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0)\rangle \\
&= \frac{1}{\sin(\theta)^2}(\sin(\theta)^2\sin(\phi)^2 + \sin(\theta)^2\cos(\phi)^2) = 1 \\
\langle e_1, e_2\rangle_{f(\mathbb{R}^2)} &= \langle f_*(e_1), f_*(e_2)\rangle_{\mathbb{R}^3} = \frac{1}{\sin(\theta)}\left\langle \frac{\partial f}{\partial\theta}, \frac{\partial f}{\partial\phi} \right\rangle \\
&= \frac{1}{\sin(\theta)}\langle(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta)), \\
&\quad (-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0)\rangle = 0 \\
\langle e_2, e_1\rangle_{f(\mathbb{R}^2)} &= \langle e_1, e_2\rangle_{f(\mathbb{R}^2)} = 0,
\end{aligned}$$

hence the push-forwards indeed form an orthonormal frame, as desired. We clearly have that $w_1 = d\theta$ and $w_2 = \sin(\theta)d\phi$ is a coframe, meaning that:

$$\begin{aligned}
dw_1 &= 0 \\
dw_2 &= \cos(\theta)d\theta \wedge d\phi = -(\cos(\theta)d\phi) \wedge d\theta = -(\cos(\theta)d\phi) \wedge w_1,
\end{aligned}$$

thus since $dw_1 = w_{12} \wedge w_2$ and $dw_2 = -w_{12} \wedge w_1$, we have:

$$\begin{aligned}
w_{12} &= \cos(\theta)d\phi \\
dw_{12} &= -\sin(\theta)d\theta \wedge d\phi = -w_1 \wedge w_2
\end{aligned}$$

so $K = 1$.

We now consider $\tilde{e}_3 = f_*(e_1) \times f_*(e_2) = (\cos(\phi)\sin(\theta), \sin(\phi)\cos(\theta), \cos(\theta))$, the cross product of our frame (recall that we denote $\tilde{e}_1 = f_*(e_1)$ and $\tilde{e}_2 = f_*(e_2)$). We have:

$$\begin{aligned} d\tilde{e}_3 &= \frac{\partial \tilde{e}_3}{\partial \theta} d\theta + \frac{\partial \tilde{e}_3}{\partial \phi} d\phi = (\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta))d\theta \\ &+ (-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0)d\phi = d\theta\tilde{e}_1 + \sin(\theta)d\phi\tilde{e}_2, \end{aligned}$$

and since $d\tilde{e}_3 = w_{31}\tilde{e}_1 + w_{32}\tilde{e}_2$, we find that $w_{13} = -w_{31} = -d\theta$ and $w_{23} = -w_{32} = -\sin(\theta)d\phi$. We can compute:

$$w_{13} \wedge w_2 + w_1 \wedge w_{23} = (-d\theta) \wedge (\sin(\theta)d\phi) + d\theta \wedge (-\sin(\theta)d\phi) = -2w_1 \wedge w_2,$$

thus since we know that $2Hw_1 \wedge w_2 = w_{13} \wedge w_2 + w_1 \wedge w_{23}$, we have that $H = -1$.

Finally, using that $\mathbb{S}^2$ is symmetric, we can compute this at every single point provided we rotate the sphere so that our point of interest lies where our frame is well defined. Hence we always have $K = 1$ and $H = -1$.

Exercise 5

Suppose $h, g : \mathbb{R} \longrightarrow \mathbb{R}$ are two smooth functions satisfying $h \neq 0$ and $(\partial h/\partial s)^2 + (\partial g/\partial s)^2 = 1$. Let $U = \{(s,v) \in \mathbb{R}^2 : s \in \mathbb{R},\ v \in (0, 2\pi)\}$ and $x : U \longrightarrow \mathbb{R}^3$ given by $x(s,v) = (h(s)\cos(v), h(s)\sin(v), g(s))$.

To compute the curvature we consider the frame in U given by $e_1 = \partial/\partial s$ and $e_2 = (1/h(s))\partial/\partial v$, defined in the whole U. From now on, we omit the variables where we consider h to not overload the notation. To prove that it is orthonormal in $x(U)$ with the induced metric, we compute:

$$\begin{aligned}
\langle e_1, e_1\rangle_{x(U)} &= \langle x_*(e_1), x_*(e_1)\rangle_{\mathbb{R}^3} = \left\langle \frac{\partial x}{\partial s}, \frac{\partial x}{\partial s}\right\rangle \\
&= \left\langle \left(\frac{\partial h}{\partial s}\cos(v), \frac{\partial h}{\partial s}\sin(v), \frac{\partial g}{\partial s}\right), \left(\frac{\partial h}{\partial s}\cos(v), \frac{\partial h}{\partial s}\sin(v), \frac{\partial g}{\partial s}\right)\right\rangle \\
&= \left(\frac{\partial h}{\partial s}\right)^2 \cos(v)^2 + \left(\frac{\partial h}{\partial s}\right)^2 \sin(v)^2 + \left(\frac{\partial g}{\partial s}\right)^2 = 1 \\
\langle e_2, e_2\rangle_{x(U)} &= \langle x_*(e_2), x_*(e_2)\rangle_{\mathbb{R}^3} = \frac{1}{h^2}\left\langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v}\right\rangle \\
&= \frac{1}{h^2}\langle(-h\sin(v), h\cos(v), 0), (-h\sin(v), h\cos(v), 0)\rangle \\
&= \frac{1}{h^2}(h^2\sin(v)^2\sin(\phi)^2 + h^2\cos(v)^2) = 1 \\
\langle e_1, e_2\rangle_{x(U)} &= \langle f_*(e_1), f_*(e_2)\rangle_{\mathbb{R}^3} = \frac{1}{h}\left\langle \frac{\partial x}{\partial s}, \frac{\partial x}{\partial v}\right\rangle \\
&= \frac{1}{h}\left\langle \left(\frac{\partial h}{\partial s}\cos(v), \frac{\partial h}{\partial s}\sin(v), \frac{\partial g}{\partial s}\right), (-h\sin(v), h\cos(v), 0)\right\rangle = 0 \\
\langle e_2, e_1\rangle_{x(U)} &= \langle e_1, e_2\rangle_{x(U)} = 0,
\end{aligned}$$

hence the push-forwards indeed form an orthonormal frame, as desired. We clearly have that $w_1 = ds$ and $w_2 = hdv$ is a coframe, meaning that:

$$\begin{aligned}
dw_1 &= 0 \\
dw_2 &= \frac{\partial h}{\partial s} d\theta \wedge dv = -\left(\frac{\partial h}{\partial s} dv\right) \wedge ds = -\left(\frac{\partial h}{\partial s} dv\right) \wedge w_1,
\end{aligned}$$

thus since $dw_1 = w_{12} \wedge w_2$ and $dw_2 = -w_{12} \wedge w_1$, we have:

$$\begin{aligned}
w_{12} &= \frac{\partial h}{\partial s} dv \\
dw_{12} &= \frac{\partial^2 h}{\partial s^2} ds \wedge dv = \frac{\partial^2 h}{\partial s^2}\frac{1}{h} ds \wedge (hdv) = -\left(-\frac{\partial^2 h}{\partial s^2}\frac{1}{h}\right) w_1 \wedge w_2
\end{aligned}$$

so indeed:

$$K = -\frac{\partial^2 h}{\partial s^2}\frac{1}{h} = \frac{h''}{h}.$$

References

[1] M. Spivak, *A Comprehensive Introduction to Differential Geometry - Volume 1*, Publish or Perish INC., 2005.

[2] J. M. Lee, *Introduction to Smooth Manifolds (Second Edition)*, Springer-Verlag, 2013.