Topology I - Homework 1

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Exercise 1.1

1. In X the set of all right-infinite binary words we define dist : $X \times X \longrightarrow \mathbb{R}$ as $\text{dist}(x, y) = e^{-|x\wedge y|}$ (with $|x \wedge y|$ being the length of the longest common prefix of x and y) when $x \neq y$ and as $\text{dist}(x, y) = 0$ when $x = y$. We want to prove that this is a metric on X. Take $x, y, z \in X$ different, then:

   (a) By definition, $\text{dist}(x, y) \geq 0$ since it is an exponential, and $\text{dist}(x, y) = 0$ only when (thus implying) $x = y$.

   (b) Symmetry: we have $\text{dist}(x, y) = e^{-|x\wedge y|} = e^{-|y\wedge x|} = \text{dist}(y, x)$ (if $x = y$ this is obvious).

   (c) Triangle inequality: we will abuse the fact that for any $a, b \in \mathbb{R}$ with $a \geq b$ then $e^{-a} \geq e^{-b}$, having equality in the second one only when we have it in the first one. First, suppose $|x\wedge z| \geq |x\wedge y|$, then we have $|z\wedge y| = |x\wedge y|$ and thus $\text{dist}(x, y) = e^{-|x\wedge y|} < e^{-|x\wedge z|}+e^{-|x\wedge y|} = \text{dist}(x, z)+\text{dist}(z, y)$ (if $|z\wedge y| \geq |x \wedge y|$ the argument is symmetric). Second, suppose $|x \wedge z| < |x \wedge y|$, then $|y\wedge z| < |x\wedge y|$ thus both $e^{-|x\wedge y|} < e^{-|x\wedge z|}$ and $e^{-|x\wedge y|} < e^{-|z\wedge y|}$ hold, meaning that $\text{dist}(x, y) = e^{-|x\wedge y|} < 2e^{-|x\wedge y|} < e^{-|x\wedge z|} + e^{-|z\wedge y|} = \text{dist}(x, z) + \text{dist}(z, y)$ (if $|z \wedge y| < |x \wedge y|$ the argument is symmetric).

      Note that since we are taking different elements in X, the triangle inequality is strict in both cases (this will be useful for later).

2. Let (Y, d) be a metric space with d being an ultrametric, $x, y, z \in Y$ distinct. Then $d(x, y) \leq \max(d(x, z), d(z, y)) < d(x, z) + d(z, y)$ since all distances are positive numbers.

3. Let (Y, d) be a metric space with d being an ultrametric, we want to prove that if two balls in Y intersect nontrivially, then one of them is inside the other. Take $x, y \in Y$ and $B_\varepsilon(x)$, $B_\delta(y)$ two balls with $B_\varepsilon(x) \cap B_\delta(y) \neq \emptyset$. Assume $\varepsilon \geq \delta$, we prove that $B_\delta(y) \subset B_\varepsilon(x)$. Take $s \in B_\varepsilon(x) \cap B_\delta(y)$, we have $d(s, y) < \delta$ and $d(s, x) < \varepsilon$. Take $z \in B_\delta(y)$, that is $d(z, y) < \delta$, now $d(z, s) \leq \max(d(z, y), d(s, y)) < \delta$ and thus $d(z, x) \leq \max(d(z, s), d(s, x)) < \max(\delta, \varepsilon) = \varepsilon$, meaning that $z \in B_\varepsilon(x)$.

4. We want to see that dist as above is an ultrametric on X, but this is a consequence of the proof we used to prove the triangle inequality. First if $|x \wedge z| \geq |x \wedge y|$, then we have $|z \wedge y| = |x \wedge y|$ and thus both $e^{-|x\wedge y|} < e^{-|x\wedge z|}$ and $e^{-|x\wedge y|} = e^{-|z\wedge y|}$ thus $\text{dist}(x, y) \leq \max(\text{dist}(x, z), \text{dist}(z, y))$ (if $|z \wedge y| \geq |x \wedge y|$ the argument is symmetric). Second if $|x \wedge z| < |x \wedge y|$, then $|y \wedge z| < |x \wedge y|$ thus both $e^{-|x\wedge y|} < e^{-|x\wedge z|}$ and $e^{-|x\wedge y|} < e^{-|z\wedge y|}$ hold, meaning that $\text{dist}(x, y) \leq \max(\text{dist}(x, z), \text{dist}(z, y))$ (if $|z \wedge y| < |x \wedge y|$ the argument is symmetric, and in fact, this is a strict inequality).

Exercise 1.2

We define on a metric space (X,d) two functions by when $x,y \in X$ then $\overline{d}(x,y) = \min(1, d(x,y))$ and $d'(x,y) = d(x,y)/(1+d(x,y))$.

1. We show that $\overline{d}$ and d' are both metrics, that is, for $x,y,z \in X$ distinct, we must have in $\overline{d}$:

   (a) $\overline{d}(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$ since d is a metric.

   (b) Symmetry: if $\overline{d}(x,y) = 1$ then $1 \leq d(x,y) = d(y,x)$ and thus $d(y,x) = 1$, if $\overline{d}(x,y) = d(x,y) = d(y,z) = \overline{d}(y,z)$.

   (c) Triangle inequality: we know that $d(x,z) \leq d(x,y) + d(y,z)$. Consider $\overline{d}(x,z)$ and all the possible sums of $\overline{d}(x,y) + \overline{d}(y,z)$, which are 2, $1 + d(y,z)$, $d(x,y) + 1$, $d(x,y) + d(y,z)$. If $\overline{d}(x,z) = 1$ (that is $d(x,z) \geq 1$) then obviously $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$ since in the right hand side we always have one plus a positive number, except in the last case when $\overline{d}(x,z) = 1 \leq d(x,z) \leq d(x,y) + d(y,z)$ and it also follows. If $\overline{d}(x,z) < 1$ (that is $d(x,z) < 1$) then again $\overline{d}(x,z) \leq \overline{d}(x,y) + \overline{d}(y,z)$ in the first three cases since in the right hand side we always have one plus a positive number, while the last case is clear by the triangle inequality in d.

   Which is obviously bounded by 1. And in d':

   (a) $d'(x,y) = 0$ if and only if $d(x,y) = 0$ if and only if $x = y$ since d is a metric.

   (b) $d'(x,y) = d(x,y)/(1+d(x,y)) = d(y,x)/(1+d(y,x)) = d'(y,z)$.

   (c) We have that $d'(x,z) = d(x,z)/(1+d(x,z)) \leq (d(x,y)+d(y,z))/(1+d(x,y)+d(y,z)) = d(x,z)/(1+d(x,y)+d(y,z)) + d(y,z)/(1+d(x,y)+d(y,z)) \leq d(x,y)/(1+d(x,y)) + d(y,z)/(1+d(y,z)) = d'(x,y) + d'(y,z)$, the first inequality being true since $d(x,z) + d(x,z)d(x,y) + d(x,z)d(y,z) \leq d(x,y) + d(y,z) + d(x,z)d(x,y) + d(x,z)d(y,z)$ by the triangle inequality of d and the second inequality being true since we decrease the denominator in both fractions.

   Which is obviously also bounded by 1.

2. We want to see that the topologies on X induced by d, $\overline{d}$ and d', say τ, $\overline{\tau}$ and τ', are the same. For this, we will prove that $\tau \subset \overline{\tau} \subset \tau' \subset \tau$ (it is enough to prove this for balls since they form a basis of the topology).

   (a) $\tau \subset \overline{\tau}$: take $B_\varepsilon^d(x) \in \tau$. Note that we can make ε as small as we want (balls are open). Thus when $\varepsilon < 1$, we have for every $y \in B_\varepsilon^d(x)$ that $d(y,x) < \varepsilon$, and that $\overline{d}(y,x) = d(y,x) < \varepsilon$, thus $y \in B_\varepsilon^{\overline{d}}(x) \in \overline{\tau}$ and $B_\varepsilon^{\overline{d}}(x) \subset B_\varepsilon^d(x)$.

(b) $\overline{\tau} \subset \tau'$: take $B_\varepsilon^{\overline{d}}(x) \in \overline{\tau}$. We can again make ε as small as we want, thus when $\varepsilon < 1$ we have for every $y \in B_\varepsilon^{\overline{d}}(x)$ that $d(y,x) = \overline{d}(y,z) < \varepsilon$. Thus $d'(y,x) = d(y,x)/(1+d(y,x)) < \varepsilon/(1+\varepsilon)$, meaning that taking $\delta = \varepsilon/(1+\varepsilon)$ we have $y \in B_\delta^{d'}(x) \in \tau'$ and $B_\delta^{d'}(x) \subset B_\varepsilon^{\overline{d}}(x)$.

(c) $\tau' \subset \tau$: take $B_\varepsilon^{d'}(x) \in \tau'$. We have for every $y \in B_\varepsilon^{d'}(x)$ that $d(y,x)/(1+d(y,x)) = d'(y,x) < \varepsilon$. We can again make ε as small as we want, thus when $\varepsilon < 1$ we have $d(x,y) < \varepsilon/(1-\varepsilon)$ well defined, and setting $\delta = \varepsilon/(1-\varepsilon)$, we obtain that $y \in B_\delta^{d}(x) \in \tau$ and $B_\delta^{d}(x) \subset B_\varepsilon^{d'}(x)$.

Exercise 2.1

We say that $A \subset \mathbb{Z}$ is a symmetric subset when for every $x \in A$ we have $-x \in A$. Consider τ_S the set containing all symmetric subsets of $\mathbb{Z}$.

1. Prove that τ_S is a topology. We just check that the three conditions are satisfied:

   (a) $Z \in \tau_S$ and $\emptyset \in \tau_S$ obvoiusly (the condition on the empty set is empty, and thus it is symmetric).

   (b) Given $\{U_j\}_{j\in J}$ with $U_j \in \tau_S$ for every $j \in J$, consider $x \in \bigcup_{j\in J} U_j$. This means that $x \in U_j$ for certain $j \in J$ and thus $-x \in U_j$ (since $U_j \in \tau_S$) and thus $-x \in \bigcup_{j\in J} U_j$, that is, $\bigcup_{j\in J} U_j \in \tau_S$.

   (c) Given $U, V \in \tau_S$, consider $x \in U \cap V$. This means that $x \in U$ and $x \in V$ and thus $-x \in U$ and $-x \in V$ (by hypothesis), meaning that $-x \in U \cap V$, that is, $U \cap V \in \tau_S$.

2. Prove that $(\mathbb{Z}, \tau_S)$ is a second countable space, that is, it admits a countable basis. We simply have to consider the set $\mathcal{B} = \{\{-x, x\} : n \in \mathbb{Z}\}$. This is a basis since given $U \in \tau_S$ we have $U = \bigcup_{x\in U} \{-x, x\}$ (because U is symmetric), and it is countable since by definition the map $f : \mathbb{Z} \longrightarrow \mathcal{B}$ given by $f(x) = \{-x, x\}$ is a bijection, and $\mathbb{Z}$ is countable.

Exercise 2.2

Let X be a topological space with basis $\mathcal{B}$ and $A \subset X$. Show that $\mathcal{B}_A = \{A \cap B : B \in \mathcal{B}\}$ is a basis for the subspace topology on A.

We know that the subspace topology is $\tau_A = \{A \cap U : U \text{ open in } X\}$. Take $A \cap U \in \tau_A$. Since U is open, we can express $U = \bigcup_{i \in I} B_i$ for certain $B_i \in \mathcal{B}$ for every $i \in I$. Now $A \cap U = A \cap \left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} (A \cap B_i)$, where $A \cap B_i \in \mathcal{B}_A$ for every $i \in I$. Thus this is an expression of $A \cap U$ as union of elements of $\mathcal{B}_A$, that is, $\mathcal{B}_A$ is a basis for the subspace topology.