Topology I - Homework 2

Pablo Sánchez Ocal

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Exercise 3.1

Given X and Y topological spaces with $X = \bigcup_{j \in J} A_j$ and $f_j : A_j \longrightarrow Y$ for $j \in J$ a family of continuous functions such that coincide over all intersections of the cover, define $f : X \longrightarrow Y$ by setting $f(x) = f_j(x)$ for $x \in A_j$ (this is well defined by the condition above).

First, notice that for any open $U \in Y$ we have $f^{-1}(U) = \bigcup_{i \in J} f_j^{-1}(U) \cap A_j$ by definition of f. Moreover, $f_j^{-1}(U)$ is open for every $j \in J$ by continuity of the functions.

Show that f is continuous if either of the following is true:

1. If each A_j, $j \in J$ is open, notice that for any open $U \in Y$ we have $f^{-1}(U) = \bigcup_{i \in J} f_j^{-1}(U) \cap A_j$ by definition of f. Moreover, $f_j^{-1}(U)$ is open for every $j \in J$ by continuity of the functions. Thus $f_j^{-1}(U) \cap A_j$ is open, and since the union of opens is open, $f^{-1}(U)$ is open thus f is continuous.

2. If each A_j, $j \in J$ is closed and J is finite, notice that for any closed $T \in Y$ we have $f^{-1}(T) = \bigcup_{i \in J} f_j^{-1}(T) \cap A_j$ by definition of f. Moreover, $f_j^{-1}(T)$ is closed for every $j \in J$ by continuity of the functions. Thus $f_j^{-1}(T) \cap A_j$ is closed, and since a finite union of closed is closed, $f^{-1}(T)$ is closed thus f is continuous.

Exercise 3.3

Let X and Y be topological spaces, $f : X \longrightarrow Y$ a function. Prove that:

1. f is continuous if and only if, for all $A \subset X$ we have $f(\overline{A}) \subset \overline{f(A)}$:

   $\Longrightarrow$) Take $A \subset X$, by definition: $\overline{f(A)} = \bigcap_{f(A) \subset T} T$ with all T closed. Now when $f(A) \subset T$ we have $A \subset f^{-1}(f(A)) \subset f^{-1}(T)$ and thus $\overline{A} \subset \overline{f^{-1}(T)} = f^{-1}(T)$ since f is continuous. Thus $f(\overline{A}) \subset f(f^{-1}(T)) \subset T$, and since this is for every T, $f(\overline{A}) \subset \overline{f(A)}$.

   $\Longleftarrow$) Consider $T \subset Y$ closed. By hypothesis, we have $f(\overline{f^{-1}(T)}) \subset \overline{f(f^{-1}(T))} \subset \overline{T} = T$. This means that $\overline{f^{-1}(T)} \subset f^{-1}(T)$, and since $f^{-1}(T) \subset \overline{f^{-1}(T)}$ always, we have the equality, meaning that $f^{-1}(T)$ is closed in X, thus f is continuous.

2. f is closed if and only if, for all $A \subset X$ we have $f(\overline{A}) \supset \overline{f(A)}$.

   $\Longrightarrow$) Let $A \subset X$, we have that $f(\overline{A})$ is closed because $\overline{A}$ and f are closed. Again by definition $\overline{f(A)} = \bigcap_{f(A) \subset T} T$ for T closed and thus since $A \subset \overline{A}$ we have $f(A) \subset f(\overline{A})$, in particular $\overline{f(A)} \subset f(\overline{A})$.

   $\Longleftarrow$) Take $T \subset X$ closed, by hypothesis and since $\overline{T} = T$ we have $f(T) \subset \overline{f(T)} \subset f(\overline{T}) \subset f(T)$. Thus $f(T) = \overline{f(T)}$ is closed, and f is closed.

Exercise 3.5

Consider $\mathbb{Z}$ with the symmetric topology τ_S. Consider $A = \{-2, -1, 0, 1, 2, 3\}$. We notice that in τ_S every open is closed (and every closed is open).

1. $\overline{A} = \{-3, -2, -1, 0, 1, 2, 3\}$, the smallest closed containing A.

2. $Int(A) = \{-2, -1, 0, 1, 2\}$, the biggest open in A.

3. The limit points are $L = \{-3, -2, -1, 1, 2\}$, since $-3 \in \{-3, 3\}$ with $3 \in A$, $-2 \in \{-2, 2\}$ with $2 \in A$, $2 \in \{-2, 2\}$ with $-2 \in A$, $-1 \in \{-1, 1\}$ with $1 \in A$, $1 \in \{-1, 1\}$ with $-1 \in A$, but $0 \in \{0\}$ and $3 \in \{-3, 3\}$ have no element different than themselves when intersecting A.

Exercise 4.1

Let X be a finite complement space ($\tau = \{U : U^C \text{ finite}\} \cup \{\emptyset\}$) with $|X| \geq 2$. We want to see that X is connected if and only if X is infinite.

$\Longrightarrow$) Suppose X finite. Then X is not connected since an open U has U^C finite thus U itself is finite, meaning that U is both open and closed (we can pick both U, U^C non empty since $|X| \geq 2$).

$\Longleftarrow$) Suppose X is not connected, that is, we can write $X = A \cup B$ with both A, B non empty opens (that we can pick since $|X| \geq 2$) with empty intersection. Then by the topology A is finite and B is finite, meaning that X is finite.

Exercise 4.2

We want to show that the Cantor set is totally disconnected. We will do more, since the Cantor set is equipped with an ultrametric, we will prove that any metric spaces with an ultrametric are totally disconnected, thus in particular the Cantor set is.

We will use that for an ultrametric we have for any $x, y \in X$ that $d(x,y) \leq \max(d(x,z), d(z,y))$. Take $x \in X$, consider the ball $B_r(x)$ (meaning that $t \in B_r(x)$ when $d(x,t) < r$). We claim that $B_r(x)$ is open and closed: take $y \in X \setminus B_r(x)$, we have $d(x,y) \geq r$, we will prove that $X \setminus B_r(x)$ is open. As desired, we have $B_r(y) \subset X \setminus B_r(x)$ (i.e. $B_r(x) \cap B_r(y) = \emptyset$) since if $z \in B_r(x) \cap B_r(y)$ then $d(x,y) \leq \max(d(x,z), d(z,y)) < r$ because $d(x,z) < r$ and $d(z,y) < r$. This is a contradiction, meaning that we indeed have $B_r(x) \cap B_r(y) = \emptyset$ and $B_r(x)$ is closed.

Now, since we can write for any $x \in X$ that $\{x\} = \bigcap_{r>0} B_r(x)$ and any intersection of closed sets is closed, we have that $\{x\}$ is closed, thus X is totally disconnected.