Topology I - Homework 3

Pablo Sánchez Ocal

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Exercise 5.1

Let X be a topological space, $A \subset X$ and x is a limit point of A (every neighborhood of x contains a point from $A \setminus \{x\}$).

1. Show that if X is T_1, then every neighborhood of x contains infinitely many points from A: suppose there is an open neighborhood U of x with $(A \setminus \{x\}) \cap U$ finite, say $(A \setminus \{x\}) \cap U = \{y_1, \dots, y_n\}$. Then, since X is T_1, for every $y_i \in (A \setminus \{x\}) \cap U$ there exists $U_{y_i} \ni x$ open with $y_i \notin U_{y_i}$, and thus $x \in U \cap U_{y_i}$ is an open neighborhood but $y \notin U \cap U_{y_i}$. This means that $x \in U \cap U_{y_1} \cap \dots \cap U_{y_n}$ is an open neighborhood, but $(A \setminus \{x\}) \cap U \cap U_{y_1} \cap \dots \cap U_{y_n} = \emptyset$, which would imply that x is not a limit point, a contradiction. Thus, $(A \setminus \{x\})$ must be infinite.

2. Give an example of a T_0 space X and a neighborhood of x that contains only finitely many points from A: consider $X = \{a, b\} = A$ with $\tau = \{\emptyset, \{a\}, X\}$. This is T_0 (since we have $a \in \{a\}$ $b \notin \{a\}$) and b is a limit point of A since its only neighborhood is X and $(A \setminus \{b\}) \cap X = \{a\}$, with $|A \setminus \{b\}| = |\{a\}| = 1$, finite.

Exercise 5.2

Let $f, g : X \longrightarrow Y$ be continuous functions between topological spaces.

1. Show that if Y is Hausdorff, then $X_{f=g} = \{x \in X : f(x) = g(x)\}$ is closed (in X). Equivalently, we will show that $X \setminus X_{f=g}$ is open. Take $x \in X \setminus X_{f=g}$, that is, $f(x) \neq g(x)$ in Y. Since Y is Hausdorff, there are $f(x) \in U_f$, $g(x) \in U_g$ opens with $U_f \cap U_g = \emptyset$. Now by continuity $f^{-1}(U_f)$ and $g^{-1}(U_g)$ are opens with $x \in f^{-1}(U_f) \cap g^{-1}(U_g)$ an open neighborhood of x. Moreover, if $y \in f^{-1}(U_f) \cap g^{-1}(U_g)$, then $f(y) \in U_f$ and $g(y) \in U_g$, and since they do not intersect, $f(y) \neq g(y)$ meaning that $f^{-1}(U_f) \cap g^{-1}(U_g) \subset X \setminus X_{f=g}$, and thus $X \setminus X_{f=g}$ is open, as desired.

2. Give an example in which Y is not Hausdorff and $X_{f=g}$ is not closed. Consider $X = \{a, b\} = Y$ with $\tau = \{\emptyset, \{a\}, X\}$. This is not T_1 (and thus in particular not T_2, that is Hausdorff) because X is the only open neighborhood of b and $a \in X$. Now consider:

$$\begin{array}{ccccc} f & : & X & \longrightarrow & X \\ & & a & \longmapsto & a \\ & & b & \longmapsto & b \end{array} \qquad \begin{array}{ccccc} g & : & X & \longrightarrow & X \\ & & a & \longmapsto & a \\ & & b & \longmapsto & a \end{array}$$

that are both continuous since $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{a\}) = \{a\}$, $f^{-1}(X) = X$, $g^{-1}(\emptyset) = \emptyset$, $g^{-1}(\{a\}) = X$ and $g^{-1}(X) = X$. Moreover, $f(a) = a = g(a)$, $f(b) = b \neq a = g(a)$, thus $X_{f=g} = \{a\}$ which is open but not closed, since $X \setminus \{a\} = \{b\} \notin \tau$.

Exercise 6.3

Let X be Hausdorff topologicla space, $Y \subset X$ a compact subspace and $x \in X \setminus Y$. Show that there are U and V disjoint opens with $x \in U$ and $Y \subset V$.

First, assume the above, we prove that every compact Hausdorff space is regular: take $T \subset X$ closed, $x \in X \setminus T$. Since T is closed inside Hausdorff, it is compact, thus by the above there are disjoint opens U_x, U_T with $x \in U_x$ and $T \subset U_T$, the definition of X being regular.

We will prove now the first statement. For each $y \in Y$ we have $x \neq y$, and since X is Hausdorff there are U_y, V_y disjoint opens with $x \in U_y$, $y \in V_y$. Now $Y \subset \bigcup_{y \in Y} V_y$, and since Y is compact, there are a finite number of points $y_1, \ldots, y_n$ in Y with $Y \subset \bigcup_{i=1}^n V_{y_i}$. Consider $U = \bigcap_{i=1}^n U_{y_i}$ open and $V = \cup_{y \in Y} V_y$ open. We have $x \in U$ (since it belongs to each element in the intersection), $Y \subset V$ and $U \cap V = \emptyset$ since $U \cap V_{y_i} = \emptyset$ for every $i = 1, \ldots, n$. Thus these are the opens we wanted.

Exercise 6.4

Let (X, τ) be a compact Hausdorff space, τ' and τ'' topologies on X with $\tau' \subsetneq \tau \subsetneq \tau''$.

1. Show that (X, τ') is compact: consider $\{U'_x\}_{x \in X}$ a collection of open sets in τ' with $x \in U'_x$ for every $x \in X$ (such a collection exists because $x \in X$ which is always open). Now $X = \bigcup_{x \in X} U'_x$ is an open cover with $U'_x \in \tau' \subsetneq \tau$, thus since (X, τ) is compact, there are a finite number of points $x_1, \ldots, x_n$ with $X = \bigcup_{i=1}^{n} U'_{x_i}$. However, each $U'_{x_i} \in \tau'$ for every $i = 1, \ldots, n$ because the original cover was from τ'. This means that (X, τ') is also compact.

   Show that (X, τ') is not Hausdorff: by contrapositive, suppose (X, τ') is Hausdorff. We will prove that any open $Y \in \tau$ is also $Y \in \tau'$, which is a contradiction since $\tau' \subsetneq \tau$. Since Y is open, $X \setminus Y$ is closed in (X, τ), which is compact, and thus $X \setminus Y$ is compact (in τ). By the argument above, covering $X \setminus Y$ by opens in τ', they are opens in τ (compact) and we obtain that $X \setminus Y$ is compact in τ'. Now $X \setminus Y$ is compact inside (X, τ') Hausdorff, thus closed (in τ'). This means that $Y \in \tau'$, the desired contradiction.

2. Show that (X, τ'') is Hausdorff: consider x, y two different points in X, then since (X, τ) is Hausdorff, there are disjoint opens $U \ni x$, $V \ni y$ in τ. However, since $\tau \subsetneq \tau''$, we have $U, V \in \tau''$ and thus (X, τ'') is Hausdorff.

   Show that (X, τ'') is not compact: by contrapositive, suppose (X, τ'') is compact. We will prove that any open $Y \in \tau''$ is also $Y \in \tau$, which is a contradiction since $\tau \subsetneq \tau''$. Consider $X \setminus Y$ closed in (X, τ'') (since Y is open in τ''), since (X, τ'') is compact, $X \setminus Y$ is compact. By the argument also used above, covering $X \setminus Y$ by opens in τ, they are opens in τ'' (compact) and we obtain that $X \setminus Y$ is compact in τ. Now $X \setminus Y$ is compact inside (X, τ) Hausdorff, thus closed (in τ). This means that $Y \in \tau$, the desired contradiction.