Topology I - Homework 4

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Exercise 6.5

Let $f : X \longrightarrow X$ be an injective continuous function with X compact Hausdorff. We will show that there exists a nonempty closed subset A of X with $f(A) = A$ without using injectivity. Thus we will answer the first, second (i.e. we never used injectivity) and third questions at the same time.

With all the information about X and f (except injectivity), we know that f continuous is closed since the domain is compact and the codomain is Hausdorff. Thus $f(T)$ is closed for any $T \subset X$ closed. Consider X and the images $f^n(X)$ for $n \in \mathbb{N}$. Since $f(X) \subset X$, we have the decreasing sequence:

$$X \supseteq f(X) \supseteq f^2(X) \supseteq \cdots \supseteq f^n(X) \supseteq \cdots$$

If at any point this sequence stabilizes, that is $f^n(X) = f^{n+1}(X) = f(f^n(X))$, then taking $A = f^n(X)$ (closed since f closed) we have $A = f(A)$ as desired.

Suppose this sequence never stabilizes, that is the inclusions are strict:

$$X \supsetneq f(X) \supsetneq f^2(X) \supsetneq \cdots \supsetneq f^n(X) \supsetneq \cdots$$

This means that $\mathcal{F} = \{f^n(X) : n \in \mathbb{N}\}$ is a family of closed sets with the finite intersection property, since for $n_1, \ldots, n_k \in \mathbb{N}$ with $k \in \mathbb{N}$ and $n_1 < \cdots < n_k$ we have $f^{n_1}(X) \cap \cdots \cap f^{n_k}(X) = f^{n_k}(X) \neq \emptyset$ since f is a function. Thus by compactness of X, we have that $A = \bigcap_{n=0}^{\infty} f^n(X)$ closed (since it is intersection of closed sets) is non empty. We claim that $f(A) = A$, which will show what we wanted:

$\subseteq$) $f(A) \subset \bigcap_{n=0}^{\infty} f(f^n(X)) = \bigcap_{n=1}^{\infty} f^n(X) \subset \bigcap_{n=0}^{\infty} f^n(X) = A$, since intersecting with $f^0(X) = X$ does nothing.

$\supseteq$) Let $a \in A$, we want to see that $a \in f(A)$. Consider $B = f^{-1}(a)$, which is non empty since $a \in A$ in particular means $a \in f(X)$. Consider $B_n = B \cap f^n(X)$ which is non empty since $a \in A$ in particular means $a \in f^{n+1}(X)$ hence $f(f^n(x)) = a$ for some $x \in X$ thus for $x_n = f^n(x)$ (i.e. $x_n \in f^n(X)$) we have $f(x_n) = a$ (i.e. $x_n \in B$). This means that $\mathcal{F}_B = \{B_n : n \in \mathbb{N}\}$ is a family of closed sets that has the finite intersection property for the same reason the family $\mathcal{F}$ has it, and thus by compactness $\emptyset \neq \bigcap_{n=0}^{\infty} B \cap f^n(X) \subset A$. In particular there is an element $b \in B$ in A, thus we found $b \in A$ with $f(b) = a$ hence $a \in f(A)$.

Exercise 6.7

Let A, B be compact subspaces of X a topological space.

1. Show that $A \cup B$ is compact: let $A \cup B \subset \bigcup_{j \in J} U_j$ be an open cover, now $A \subset \bigcup_{j \in J} U_j \cap A$ and $B \subset \bigcup_{j \in J} U_j \cap B$ are open covers of A and B respectively. Thus by compactness we must have $A \subset \bigcup_{j \in J_A} U_j \cap A$ and $B \subset \bigcup_{j \in J_B} U_j \cap B$ two finite open subcovers, that is, J_A and J_B are finite subsets of J. Then $A \cup B \subset \bigcup_{j \in J_A \cup J_B} U_j$ is a finite subcover since $J_A \cup J_B$ is finite, meaning that $A \cup B$ is compact.

2. Give an example of X, A and B where $A \cap B$ is not compact. Consider $\mathbb{N}$ with the discrete topology $\tau_{\mathbb{N}}$ and x, y two points not in $\mathbb{N}$. Let $X = \mathbb{N} \cup \{x, y\}$ with the topology $\tau = \tau_{\mathbb{N}} \cup \{\{x\} \cup \mathbb{N}, \{y\} \cup \mathbb{N}, \{x, y\} \cup \mathbb{N}\}$ (that is, the only opens different from the total that contain x or y are $\{x\} \cup \mathbb{N}$ and $\{y\} \cup \mathbb{N}$ respectively). This is a clearly a topology, but a fast proof is:

   (a) $\emptyset \in \tau_{\mathbb{N}} \subset \tau$, $X \in \tau$.

   (b) The unions of opens in $\tau_{\mathbb{N}}$ stay in $\tau_{\mathbb{N}}$. Whenever we have a union containing only $\{x\} \cup \mathbb{N}$ and elements of $\tau_{\mathbb{N}}$, this is just $\{x\} \cup \mathbb{N}$ (and equivalently for $\{y\} \cup \mathbb{N}$). Whenever we have both $\{x\} \cup \mathbb{N}$ and $\{y\} \cup \mathbb{N}$ or $\{x, y\} \cup \mathbb{N}$ the union is the whole X. And all of the above are open.

   (c) The finite intersections of opens in $\tau_{\mathbb{N}}$ stay in $\tau_{\mathbb{N}}$. Whenever we have a intersection containing $\{x\} \cup \mathbb{N}$, $\{y\} \cup \mathbb{N}$ or $\{x, y\} \cup \mathbb{N}$ and elements of $\tau_{\mathbb{N}}$, this is just in $\tau_{\mathbb{N}}$ (and equivalently for $\{y\} \cup \mathbb{N}$). Whenever we have only $\{x\} \cup \mathbb{N}$ with $\{y\} \cup \mathbb{N}$ the intersection is $\mathbb{N}$ (and including $\{x, y\} \cup \mathbb{N}$ does not affect anything and thus may be omitted). And all of the above are open.

   Now, any cover of $\{x\} \cup \mathbb{N}$ must contain $\{x\} \cup \mathbb{N}$ (and analogously for or $\{y\} \cup \mathbb{N}$) thus both $A = \{x\} \cup \mathbb{N}$ and $B = \{y\} \cup \mathbb{N}$ are compact. However, $A \cap B = \mathbb{N}$ is infinite and discrete, thus cannot be compact.

Exercise 7.1

Show that the Axiom of Choice is equivalent to the existence of sections for surjective functions.

$\Rightarrow$) Let J be nonempty, $\{A_j\}_{j\in J}$ a family of non empty sets, we know that there exists $f : J \longrightarrow \bigcup_{j\in J} A_j$ with $f(j) \in A_j$ for every $j \in J$.

Consider $g : X \longrightarrow Y$ a surjective function. Thus for every $y \in Y$ we have $g^{-1}(y) \neq \emptyset$. Apply the Axiom of Choice with $J = Y$ and $\{A_j\}_{j\in J} = \{A_y = g^{-1}(y)\}_{y\in Y}$ (notice $\bigcup_{y\in Y} g^{-1}(y) = g^{-1}(Y) = X$). This gives us a function $f : Y \longrightarrow X$ with $f(y) \in g^{-1}(y)$ for every $y \in Y$, meaning that $g \circ f(y) = y$ for every $y \in Y$, thus $g \circ f = 1_B$ is the identity and f is a right inverse of g.

$\Leftarrow$) Let sections of surjective functions exist, let J be non empty and $\{A_j\}_{j\in J}$ a family of non empty sets. We will need an intermediate step, for which we will use the disjoint union of sets $\bigsqcup_{j\in J} A_j$ that has for elements the pairs (a, j) with given $j \in J$ that $a \in A_j$. Notice that the function $g : \bigsqcup_{j\in J} A_j \longrightarrow J$ defined by $g(a, j) = j$ (we have $a \in A_j$) is surjective, thus there exists $\tilde{f} : J \longrightarrow \bigsqcup_{j\in J} A_j$ with $\tilde{f}(j) = (a, j)$ for certain $a \in A_j$ for every $j \in J$. Now notice that the composition $f = \iota \circ \tilde{f} : J \longrightarrow \bigcup_{j\in J} A_j$ with $\iota : \bigsqcup_{j\in J} A_j \longrightarrow \bigcup_{j\in J} A_j$ defined by $\iota(a, j) = a$ (we have $a \in A_j$) for every $j \in J$ is such that $f(j) = a \in A_j$ for every $j \in J$, thus this f is the function satisfying the Axiom of Choice that we wanted.

Notice that the naive approach of simply setting $g : \bigcup_{j\in J} A_j \longrightarrow J$ defined by $g(a) = j$ when $a \in A_j$ is not necessarily a well defined function, and if we manage to make it a function it may not be surjective:

1. We may have $A_l \cap A_k \neq \emptyset$, $l, k \in J$ with $l \neq k$ and such intersection not contained in any other element of the family. Hence the naive definition would want to send elements in the intersection to both l and k, meaning that g is not a well defined function.

2. We may want to narrow the naive definition of g to "choose" one of the indexes when the case above happens. However, this does not solve all our problems: we may have $A_j = A$ a fixed set for every $j \in J$, and since for every element $a \in \bigcup_{j\in J} A_j$ then $a \in A_{j_0}$ for a fixed $j_0 \in J$, we may "choose" to set g constant to this $j_0 \in J$ to fix the problem of definition faced above. This is at least is a function and satisfies a narrower naive definition of g, but is clearly not surjective.

This is why we need to be more careful and refine the argument and use the disjoint union of sets to properly define g so that it satisfies what we want.

Exercise 7.2

Show that Zorn's Lemma implies the existence of a basis in every vector space.

Let V be a vector space (over some field K). If $V = \{\vec{0}\}$, then $\mathcal{B} = \{\} = \emptyset$ is a basis. Suppose we have $\vec{v} \in V$ with $\vec{v} \neq \vec{0}$, notice that this means that $\{\vec{v}\}$ is linearly independent. Let $\mathcal{L}$ be the set of linearly independent subsets of V (ordered by inclusion) that contain $\{\vec{v}\}$, notice $\{\vec{v}\} \in \mathcal{L}$ and thus $\mathcal{L} \neq \emptyset$. We will use Zorn's Lemma on $\mathcal{L}$ to see that it has a maximal element $M \in \mathcal{L}$, and we will then prove that this maximal element must be a basis: it is linearly independent and it spans V (and of course is non empty since $\vec{v} \in M$ because $\{\vec{v}\} \subset M$).

Let $\{C_j\}_{j \in J}$ be a chain of elements of $\mathcal{L}$, set $C = \bigcup_{j \in J} C_j$. We claim $C \in \mathcal{L}$. Suppose not, we will achieve a contradiction. We clearly have $\vec{v} \in C$. Suppose then C is linearly dependent, there exist $\vec{v_1}, \ldots, \vec{v_n} \in C$ and $k_1, \ldots, k_n \in K \setminus \{0\}$ with $k_1\vec{v_1} + \cdots + k_n\vec{v_n} = 0$. Now we have $\vec{v_1} \in C_{j_1}, \ldots, \vec{v_n} \in C_{j_n}$ with $C_{j_k} \in \mathcal{L}$ for $1 \leq k \leq n$. Since $\{C_j\}_{j \in J}$ is a chain, we have that $C_{j_1} \cup \cdots \cup C_{j_n} = C_{j_k}$ for some $0 \leq k \leq n$ and thus we have $\vec{v_1}, \ldots, \vec{v_n} \in C_{j_k}$, meaning that C_{j_k} is linearly dependent, a contradiction since $C_{j_k} \in \mathcal{L}$. Thus C is linearly independent. This means that $\mathcal{L}$ is a partially ordered set where every chain has an upper bound, thus by Zorn's Lemma, $\mathcal{L}$ has a maximal element $M \in \mathcal{L}$.

Since $M \in \mathcal{L}$, M is linearly independent and $\vec{v} \in M$, hence we just have to show that M spans V. Suppose not, we achieve a contradiction. If there is an element $\vec{u} \in V$ such that the span of M does not contain $\vec{u}$, then $M \cup \{\vec{u}\} \in \mathcal{L}$ since $\vec{v} \in M \subset M \cup \{\vec{u}\}$ and $M \cup \{\vec{u}\}$ is linearly independent (if it were not, we could write $\vec{u}$ as a linear combination of elements in M, thus it would belong in the span of M). However, $M \subsetneq M \cup \{\vec{u}\}$, contradicting the maximality of M in $\mathcal{L}$, thus M spans V.

We found M a non empty linearly independent subset of V that spans V, thus by definition M is a basis of V.